Solving 0-sum games

If $\exists$ saddle point: easy.

e.g. $A = \begin{pmatrix} 1 & 2 \\ 0 & x \end{pmatrix}$: (row 1, col 1) is equilibrium

\[ A = \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \]
both using $\left(\frac{1}{2}, \frac{1}{2}\right)$ is equil.

Optimal replies: If $p_1$ uses strategy $x$, payoff cost vector for $p_2$ is $x^TA$

optimal response to $x$ is the minimal minimal entry of $x^TA$. If several equally small entries, pick any distribution on the small entries.
e.g. if $x^tA = (2,3,2,4)$ optimal replies are $(p, 0, 1-p, 0)$. For $p_1$: if $p_2$ uses $y$, optimal replies use maximal entries of $Ay$.

**def:** A Nash equilibrium (NE) is a pair of strategy $(x^*, y^*)$ such that $x^*$ is an optimal reply to $y^*$, and $y^*$ is an optimal reply to $x^*$.

**Thm.** For a 0-sum game, $(x, y)$ is a NE if and only if:

1. For some value $V$, all entries of $x^tA$ are $\geq V$, and if $(x^tA)_i \neq V$ then $y_i = 0$ and
2. all entries of $Ay$ are $\leq V$, and if $(Ay)_i \neq V$ then $x_i = 0$.
e.g. \( A = \begin{pmatrix} 2 & 0 & 4 \\ 1 & 3 & 3 \end{pmatrix} \) Let \( x = \left( \frac{1}{2}, \frac{1}{2} \right) \) \( y = \left( \frac{2}{4}, \frac{1}{4}, 0 \right) \).

\[
x^T A = \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ \frac{3}{2} \end{pmatrix} \quad A y = \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \end{pmatrix}
\]

\( V = \frac{3}{2} , \quad (x^T A)_2 \neq \frac{3}{2} \) and indeed \( y_3 = 0 \).

\( (x^T A)_1 \geq \frac{3}{2} \quad (A y)_1 \leq \frac{3}{2} \).

If \((x,y)\) is a NE then \( V \) is the value of \( A \).

If \( P_1 \) uses \( x \), get \( \geq V \).

If \( P_2 \) uses \( y \), pay \( \leq V \).

---

e.g. \( A = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix} \) \( P_1 : (1,0) \) or \((0,1)\) or \((p,1-p)\) for some \( p \in (0,1) \).

Case \( x = (1,0) \): \( x^T A = (1,2) \) \( y^T = (1,0) \) ... check this is NE.

Case \( x = (0,1) \): \( x^T A = (0,4) \) \( y^T = (1,0) \) \( x \) is not optimal reply to \( y \).
Case: \( x^T = (p, 1-p) \). \( x^T A = (p, 4-2p) \)

\( x \) uses both rows, so \( A y \) has equal entries.

\[ y_1 + 2y_2 = 4y_2 \quad y_1 + y_2 = 1 \quad \text{so} \quad y = \left( \frac{6}{7}, \frac{1}{7} \right) \left( \frac{2}{3}, \frac{1}{3} \right) \]

\( A y = \left( \frac{4}{3}, \frac{4}{3} \right) \). Both entries of \( y \) not 0, so both entries of \( x^T A \) are minimal, i.e. \( p = 4-2p \) or \( p = \frac{4}{3} \leq \)

**Dominating strategies**

If action \( i \) is better than \( j \), no matter what opp. does, never use \( j \).

In a 0-sum game, if rows \( i, j \) satisfy

\[ \forall k \ A_{ik} \geq A_{jk}, \text{ never pick row } k \text{ or } j \]

say row \( i \) dominates row \( j \).
Value of game unchanged by deleting row \( j \).

Similarly if \( \text{col } i \leq \text{col } j \) in every entry, can delete \( \text{col. } j \).

\[
A = \begin{pmatrix}
7 & 8 & 0 & 0 \\
2 & 11 & -10 & -82 \\
4 & 5 & 1 & 0
\end{pmatrix}
\]

row 2 dominated by 1.

\[
\begin{pmatrix}
7 & 8 & 0 & 0 \\
4 & 5 & 1
\end{pmatrix}
\]

col. 1,2,3 dominated by 4.

reduce to \((\emptyset)\) reduce: \((\emptyset)\)
A 2xn game is not too bad.

\[ V = \max \min_{x \in \Delta^n} x^T A y \]

\[ \min_{y \in \Delta^n} \]

Reduces to finding the max of the min of several linear funcs.