Recall: \( A \) is an \( m \times n \) matrix.

Pl. 1 picks a row \( i \)
Pl. 2 picks a col. \( j \)
Pl. 2 pays \( A_{ij} \) to Pl. 1

E.g. \( A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \)

Safety strategies: pick a row with largest min. col. and smallest max.

Saddle point: the two are equal.

\[
\begin{pmatrix} \\
7 & \\
& 7 \\
& & \leq 7 \\
& & \\
& & \\
& & \\
\end{pmatrix}
\]

\( 7 \) is a saddle point
$A_{ij}$ is a saddle pt. if it is largest in row $i$ \& col $j$ and smallest in col $j$ \& row $i$.

Note: can have multiple saddle pts. but all have same value of $A_{ij}$.

**Pure / mixed strategies**

**pure**: fixed action,  
**mixed**: random action.

E.g. \( A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \) Let P1 pick each row w.p. \( \frac{1}{2} \)

If P2 pick col.1, get on average \( \frac{1}{2} \)

If row 1 w.p. \( x_1 \), \((x_1, x_2)\) is a vector, \( x^T \)

\( x_1, x_2 \geq 0 \)

\( x_1 + x_2 = 1 \) so \( x_2 = 1 - x_1 \)
Payoff vector is $x^TA = (x_1 \ x_2) \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = (x_1, 2x_2)$

Worst outcome is $\min x^TA = \max (x_1, 2x_2)$

$= \max (x_1, 2(1-x_1)) = \text{green line}$

$\min (x_1, 2(1-x_1)) = \text{red line}$

If $x_1 = \frac{2}{3}$, $x^TA = \left( \frac{2}{3}, \frac{2}{3} \right)$

P2 can pick col 1 w.p. $\frac{2}{3}$. Let $y = \left( \frac{2}{3} \right)$

$Ay = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} y = \left( \frac{2}{3} \right)$

Both players can guarantee $\frac{2}{3}$ payout.
mixed strat. for p.l. are vectors $x$ in $\mathbb{R}^m$, $x_i \geq 0$ and $\Sigma x_i = 1$

The set of such $x$ is denoted $\Delta^m$

$$\Delta^m = \{x \in \mathbb{R}^m : x_i \geq 0 \quad \Sigma x_i = 1 \}$$

$p^2$ mixed strat. are $\Delta^n$. 
\[ A = \begin{pmatrix} 40 \\ 10 \\ 15 \end{pmatrix} \quad \text{s.p.} \]

\[ A = \begin{pmatrix} 40 & 40 \\ 15 & 41 \end{pmatrix} \quad \text{no s.p.} \quad A = \begin{pmatrix} 6 & 1 \\ 3 & 7 \end{pmatrix} \]

If p.1 picks strat. \( x = \begin{pmatrix} x_1 \\ 1-x_1 \end{pmatrix} \), \( x^T A = (6x_1 + 3x_2, 1x_1 + 7x_2) \)
\[ = (6x_1 + 3(1-x_1), x_1 + 7(1-x_1)) \]
\[ = (3 + 3x_1, 7 - 6x_1) \]

\( \min(x^T A) \)
\[ x_1 = 0.429 \quad \therefore \begin{pmatrix} \frac{13}{3} \\ \frac{13}{3} \end{pmatrix} \]
For $p_2$, pick $y$, payoff vector is

$$Ay = \left(6y_1 + y_2, 3y_1 + 7y_2\right)^T = \left(1 + 5y_1, 7 - 4y_1\right)^T$$

$$\max Ay$$

Theorem (Von Neumann minimax theorem)

For any $A$, \[ \max_{x \in \Delta^m} \min_{y \in \Delta^n} x^T Ay = \min_{y \in \Delta^n} \max_{x \in \Delta^m} x^T Ay \]

$p_1$ can get this at least

$p_2$ can pay at most this.
If \( p_1 \) picks \( x \in \Delta^m \), \( p_2 \) picks \( y \in \Delta^n \), then the average outcome is \( \sum_{ij} x_i y_j A_{ij} = x^T A y \).

\( p_1 \) wants an \( x \) so that \( x^T A y \) is large for all \( y \).

i.e. an \( x \) so that \( \min_y x^T A y \) is as large as possible.

The common value is the game's value.