Monotonicity

Idea: having options is good.

Notation: \( x \xrightarrow{A} y \) if player A can move from \( x \) to \( y \).

Game: set of positions, and legal moves for \( A, B \).

Say that a game \( \hat{G} \) is better for \( A \) than \( G \) if:
when ever \( x \xrightarrow{A} y \) in \( G \), also \( x \xrightarrow{A} y \) in \( \hat{G} \).

Thm If \( \hat{G} \) is better than \( G \) for \( A \), and worse than \( G \) for \( B \) and if \( A \) wins in \( G \) from pos. \( x \), then \( A \) also wins in \( \hat{G} \) from \( x \).

Note: same set of moves: both better and worse, like \( \geq \), not \( > \).

Pf: Just follow the winning strategy for \( G \). \( \square \)
E.g. Strategy stealing in Hex:
Alice takes arbitrary \( x \), play as 2nd from here.
For Bob, this is worse game. 
(Compare: Bob first; \( G \).
Bob first, Alice has \( x (G) \).
If Alice needs to play at \( x \) to use 2nd player strategy, she plays instead \( x' \).
(As if Alice has 1 movable stone, extra valid moves to place a regular stone at \( x \), move special stone \( x \rightarrow x' \).)
0-sum games

2 player games, fully adversarial: better for A is worse outcome for B.
e.g. one wins, other loses.

Strategy: A choice of move from every possible position of a game.
e.g. Subtraction, set \{1, 2\}

Strategies eg:
* Always take 1,
  * Alternate (if remember time)
  * 2 from even 1 from odd.
* 1 unless \(n = 3n + 2\). [winning]
* Coin toss. [not deterministic]
Pure strategy = non random.
Mixed strategy = with randomness.

If A, B pick their strategies, the outcome is known. A game says what is the outcome for any pair of strategies.

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1 = Alice wins
-1 = Bob wins
0 = draw

This matrix encodes the whole game.
A 0-sum game is given by a matrix $A$. If $A$ is an $n \times m$ matrix, player I has $n$ strategies and player II has $m$ strategies. Each player picks a strategy simultaneously. If their picks are $i$ and $j$, the outcome is $A_{ij}$. Player 2 pays $A_{ij}$ to player 1.

$0$-sum: total gain is always 0.

In combinatorial games, all entries $A_{ij}$ are $\pm 1$. A winning strategy for player 1 is to pick a row of all $+1$ or a column of all $-1$. 