NIM

Several piles of chips/stones.
Valid move: take at least 1 from one pile.
Normal play: last move wins.

1 pile: N pos.
2 piles: \((a, b)\) p-pos are \(a = b\)
N-pos if \(a \neq b\)

\((1, 2, 3)\) is P-pos.

NIM-sum / Exclusive OR
ADD in base 2, no carries

Eq. \(29 \oplus 110 = 115\).
$3 \oplus 5 \oplus 6 = 0$

$(2+1) \oplus (4+1) \oplus (4+2)$

$11$
$101$
$110$
$000$

$5 \oplus 6 \oplus 7 = (1+4) \oplus (2+4) \oplus (1+2+4)$

$= 4$

**Theorem:** p-positions of NIM are exactly those with nim-sum 0.

$a \oplus b = 0$ if and only if $a = b$.

**Proof:** need to show: ① If $x$ has nim-sum 0, then every follower has nim-sum $\neq 0$.

② If $x$ has nim-sum $\neq 0$, then some follower has nim-sum 0.
0. If \( x = (x_1, \ldots, x_k) \) has nim-sum 0.

Let \( y \) be a follower, \( y_i = x_i \) for all but a single \( i \).

For that \( i \), \( y_i \neq x_i \);

some digit of \( x_i \) differs from \( y_i \), so also the

nim-sum \( \oplus y_i \neq \oplus x_i \).

2. If we replace \( x_i \) by \( x_i \oplus s \) where \( s = \oplus x_i \);

then we get a pos. y with \( \oplus y_i = 0 \).

remains to see: For some \( i \), \( x_i \oplus s \leq x_i \);

If largest digit of \( s \) is \( 2^k \), and \( 2^k \) appears in

\( x_i \) then \( x_i \oplus s \leq x_i \).