Mathematical Game theory
Assignment 9 solutions

Problem 1. A 0-sum game is played. A coin is tossed, and the payoff matrix is either $A^H$ or $A^T$ depending on the coin, with

$$A^H = \begin{pmatrix} 4 & 1 \\ 3 & 0 \end{pmatrix}, \quad A^T = \begin{pmatrix} 0 & 4 \\ 2 & 8 \end{pmatrix}.$$

(a) Suppose Player 1 gets to see the coin, and the players move simultaneously. Draw the game tree, list all possible strategies for each player, express the game in matrix form, and find the value of the game.

(b) Suppose Player 1 gets to see the coin, and announces her move before player 2 makes his choice. Repeat the tasks from part (a).

Solution:

(a) Player 1 can pick 1 or 2 in $A^H$ and 1 or 2 in $A^T$, so has 4 pure strategies: $H_1T_1, H_1T_2, H_2T_1, H_2T_2$. Player two has only 2 strategies since does not know the coin. The matrix form (in the above order for player 1) is

$$A = \begin{pmatrix} 2 & 2.5 \\ 3 & 4.5 \\ 1.5 & 2 \\ 2.5 & 4 \end{pmatrix}.$$

The strategy $H_1T_2$ dominate all others, and the value is 3. This is also seen directly in this game, since no matter what Player 2 does, in $A^H$ player 1 picks row 1, and in $A^T$ row 2, so that’s their optimal strategy.

(b) Player 1 has the same strategies, but player 2 gets to react to player 1’s move so has 4 pure strategies: 11, 12, 21, 22, where $ij$ means if player one picks 1 you pick $i$, and if player 1 picks 2 you pick $j$. The matrix now is

$$A = \begin{pmatrix} 2 & 4 & 2.5 & 2.5 \\ 3 & 6 & 1.5 & 4.5 \\ 1.5 & 0 & 3.5 & 2 \\ 2.5 & 4 & 2.5 & 4 \end{pmatrix}.$$

The value of this is 2.5, with player 1 picking $H_2T_2$ and player 2 a mixture of 11 and 21. Specifically, if player 1 picks row 1 player 2 picks randomly, whereas against row 2 he picks column 1.

Problem 2. There are $n$ roads between $s$ and $t$, with the $i$’th road having latency function $\ell_i(x) = 1 + ix$. Find the optimal flow for 1 unit of traffic from $s$ to $t$. Find the Nash equilibrium and price of anarchy.
**Solution:** Assume the flow through road $i$ is $F_i$. Then the average latency is
\[
\sum F_i(1 + iF_i) = \sum F_i + \sum iF_i^2 = 1 + \sum iF_i^2.
\]
Minimizing this with $\sum F_i = 1$ using Lagrange multipliers we have $2iF_i = \lambda$, so $F_i = \frac{i}{\lambda}$ for some $c$. Since $\sum F_i = 1$, we find $c^{-1} = \sum_{i=1}^{n} \frac{1}{i}$.

The Nash equilibrium has $1 + iF_i = c$, so $F_i = \frac{c-1}{i}$ for some $c$, which gives the exact same flow, and so the price of anarchy is 1.

**Problem 3.** There are two roads between $s$ and $t$. The first has latency function $\ell_1(x) = 1 + x$. The second has latency function $\ell_2(x) = x^2$. Suppose the total amount traffic is $A$. Find the latency in the optimal flow, and the price of anarchy.

**Problem 4.** Consider a $3 \times 3$ grid, with a unit amount of traffic from the bottom left corner $(0,0)$ to the top right $(2,2)$. All roads are one-way to the right and upward. Find the equilibrium and optimal flows and price of anarchy in the following settings.
(a) The latency function for each horizontal is $\ell_e(x) = ax$ and for vertical edges $\ell(x) = bx$ for some $a, b > 0$.
(b) The latency function for each edge is $\ell_e(x) = 1 + x$.
(c) The latency function for each horizontal is $\ell_e(x) = 1$ and for vertical edges $\ell(x) = x$.

**Problem 5.** In an $n \times n$ grid of one-way edges, with latency functions $\ell_e(x) = x$, find the optimal flow from $(0,0)$ to $(n,n)$.

**Problem 6.** Consider a traffic network form moving a unit flow from a source $s$ to a target $t$. Suppose the latency functions are all of the form $\ell_e(x) = a_e x$, where $a_e > 0$ is a constant for each edge. Show that the price of anarchy in such a network is $1$. (Hint: this is similar to a proof from the next class, using the inequality $xy \leq (x^2 + y^2)/2$.)