Problem 1. Analyse 3-card Kuhn poker (problem 6.4 from Peres-Karlin).

• There are 3 cards: Jack, Queen and King.
• Each player pays $1 into the pot.
• Each player is dealt one of the cards.
• Player I can either pass (P) or bet (B) $1.
  – If player I bets, then player II can either fold (F) or call (C) (adding $1 to the pot).
  – If player I passes, then player II can pass (P) or bet $2 (B). If player II bets, then player I can either fold or call (match the bet).
• If one of the players folds, the other player takes the pot. If neither folds, the player with the high card wins what’s in the pot.

Find a Nash equilibrium in this game via reduction to the normal (matrix) form. Observe that in this equilibrium, there is bluffing and overbidding.

Solution: For each card player I has they can either bet, or pass and if player 2 bets call, or pass and fold. denote these by B, C, and F. Thus a pure strategy is a triplet of such choices for J, Q, K. For example, we can use FBB for the strategy where with the J we pass then fold, and with the Q or K we bet. This gives $3^3$ strategies. However, with the J is is always better to fold than to call, and with the K better to call than fold, so after removing dominated strategies we have

$$BBB, BBC, BCB, BCC, BFB, BFC, FBB, FBC, FCB, FCC, FFB, FFC.$$  

For player 2, they can either call (C) or fold (F) after a bet, and either bet (B) or pass (P) after a pass. This gives $4^3 = 64$ strategies, but the same domination leaves 16 pure strategies. For example PFPCBC means to call with Q or K and bet only with the K.

With a given pair of strategies, we can find the outcome with each dealt cards, and the expectation. The strategies FBB and PFPCBC give:

- (J, Q): P1 passes, P2 passes, result is -1 for P1.
- (J, K): pass, bet, fold, result is -1
- (Q, J): bet, fold, +1
- (Q, K): bet, call, -2
- (K, J): bet, fold, +1
- (K, Q): bet, call, +2.

Average for these strategies is 0.
Computing the $12 \times 16$ grid, and removing some dominated strategies leaves
\[
\begin{bmatrix}
2 & -1 & 0 & -3 \\
3 & 0 & -1 & -4 \\
-1 & -3 & 1 & -1 \\
1 & -2 & 1 & -2 \\
1 & 1 & -1 & -1 \\
2 & 2 & -2 & -2 \\
-2 & -1 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
This has value 0, with the strategies: last row for player 1 and $(0,.5,0,.5)$ for player 2.

**Problem 2.** Consider the game from class:
- A coin is tossed, and Alice sees the result.
- There are $T$ rounds (with the same coin).
- In each round, Alice picks H or T and Bob picks H or T, and their choices are revealed.
- At the end of $T$ rounds, Bob pays Alice 1 for each round where they both picked the same as the hidden coin.

As noted in class, the value for $T = 1$ is $1/2$ and for $T = 2$ it is $3/4$. What happens for 3 rounds?

(a) Show that Alice can guarantee getting at least 1 on average.

(b) Find a strategy for Bob where he pays at most 1 on average.

**Solution:**

(a) Let Alice ignore the information and play randomly in rounds 1 and 2, and pick the hidden coin in round 3. She receives an average payoff of $1/4$ in the first two rounds and $1/2$ in the last (no matter what Bob picks), for a total of 1.

(b) Let Bob play randomly in rounds 1 and 2. In round 3, if Alice’s first two moves are HH, Bob picks T. If Alice’s first two moves are TT, Bob picks H. Otherwise Bob plays randomly. Suppose the hidden coin is H. If Alice picks HHH, Bob pays $1/2 + 1/2 + 0$. If Alice picks HTH or THT, Bob pays $1/2 + 0 + 1/2$. If Alice picks TTH, Bob pays $0 + 0 + 1$. The total is 1 in every case.

**Problem 3** (bonus). What happens for higher $T$ in the previous problem? Let $V_T$ be the value. A full solution is to calculate the value with proof. Partial credit for non-matching lower and upper bounds on the value. The following are some directions to consider.

(a) What is $V_4$? Compute the value if you can. Give upper and lower bounds if not.

(b) Prove that $V_T \geq V_{T-1} + \frac{1}{2}$, and therefore $V_T \geq \frac{T+1}{4}$.

(c) Prove that for $T$ even, $V_T \leq \frac{3T}{8}$.

(d) Prove that $\limsup \frac{V_T}{T} \leq \frac{1}{3}$.

(e) Prove that $\lim \frac{V_T}{T} = \frac{1}{4}$. 
Solution:
(a) It turns out that $V_4 = 9/7$.
This is a 0-sum game, so the value can be computed by the minimax theorem. Bob’s possible pure strategies are to pick H or T after each sequence observed from Alice. For Bob’s $k$’th move there are $2^{k-1}$ possible observed moves from Alice, and Bob needs to pick H or T after each of these, which gives $2^{1+2+3+...+2^{T-1}} = 2^{2T-1}$ pure strategies for Bob. For $T = 4$ this are $2^{15}$ pure strategies. For Alice, there is also the hidden coin to consider, so there are $2^{2T+1} - 2$ pure strategies.
While the space of strategies is large, it can be reduced significantly. A key idea is that Bob’s actions do not give Alice any new information, so the value does not change if Alice ignores Bob’s choices. (More precisely, for any strategy that Bob picks, any pure strategy of Alice is equivalent to a mixture of strategies that ignore Bob’s actions. Therefore we can (and do) assume that for each value of the hidden coin, Alice has $2^T$ pure strategies, giving the sequence of choices she makes. We can also use symmetry to assume that the distribution Alice uses if the coin is H is the opposite of the distribution if it is T.
Next, Bob can restrict to behavioural strategies (see Chapter 3). This means after each observed sequence from Alice there is a probability of picking H. Thus instead of $2^{15}$ probabilities for $T = 4$ there are only 15 probabilities for a mixed strategy for Bob. By symmetry, we can assume that if bob picks H with probability $x_a$ after seeing a sequence $a$, then $x_{a^c} = 1 - x_a$, where $a^c$ is the complementary sequence. This also means in the first step Bob plays H with probability $1/2$, leaving only 7 parameters. Given such a strategy, we check which of Alice’s pure strategies give the maximal value, and minimize this over Bob’s strategies. We can now use convex optimization on 7 variables to find the value of the game.
The function is

$$f(x) = \max(x_{00}, x_0 + x_{01}, x_0 + x_{10}, x_0 + x_1 + x_{11},$$

$$3/2 - x_{11}, 5/2 - x_1 - x_{10}, 5/2 - x_0 - x_{01}, 7/2 - x_0 - x_0 - x_{00}),$$

and the minimum is attained at

$x_0 = \frac{9}{14}, x_1 = \frac{1}{2}, x_{00} = 1, x_{01} = \frac{9}{14}, x_{10} = \frac{5}{7}, x_{11} = \frac{3}{14}$

Alice receives 9/7 for any strategy, unless she picks the opposite of the coin in the first 3 rounds.
(b) If Alice ignores the hidden coin in round 1 and uses the $T$ round optimal strategy afterwards, she gets $\frac{1}{2} + V_T$, so this is a lower bound for $V_{T+1}$.
(c) This and the next are a consequence of sub-additivity: $V_{n+m} \leq V_n + V_m$, for any $m, n$. This is the case since Bob can limit Alice to $V_n$ in the first $n$ rounds, then forget what happened in those rounds and limit Alice to $V_m$ in the remaining rounds. Since $V_2 = \frac{3}{4}$, it follows that $V_{2n} \leq \frac{3n}{4}$.
(d) Since $V_3 = 1$ we get $V_{3n} \leq n$. Since $V_n$ is increasing in $n$, the claim follows.
(e) Since $V_T \geq T/4$, the limit is at least $1/4$. To get an upper bound, we need to argue that either Alice picks nearly uniform actions for almost all the time, or else she gives up too much information to Bob, and gets nothing later.