Problem 1. In the stable matching setting with \( n \) men and \( n \) women, suppose that \( M, M' \) are two stable matchings. For a man \( x \), let

\[
M(x) = a, \quad M'(x) = b, \quad M'(a) = y.
\]

Suppose \( x \) prefers \( a \) to \( b \). Show that \( a \) prefers \( y \) to \( x \).

Problem 2. Consider the stable matchings setup with women \( a_1 \ldots a_n \) and men \( b_1 \ldots b_n \). Suppose each pair \( a_i, b_j \) are either compatible or incompatible, and that every woman’s preference order is the compatible men in increasing order of index, followed by the incompatible men, and similarly for the men. (So \( a_1 \) is the top choice for all men with which she is compatible.)

Prove that there is a unique stable matching in this case.

Problem 3. The stable roommates problem (aka homosexual stable matching): There is a set of \( 2n \) people, each with a preference order over all the remaining people. A matching of the people (each matched pair will become roommates) is stable if there is no pair of people that are not matched that prefer to be roommates with each other over their assigned roommate in the matching.

Show that there are some preference orders for which there is no stable matching.

Problem 4. Consider the stable matching where every person’s preference is a uniformly random permutation. Let \( X \) be the total number of proposals made in the men-proposing Gale-Shapley algorithm. We show that \( \mathbb{E}X \leq nH_n \), where \( H_n = \sum_{i \leq n} \frac{1}{i} \) is the harmonic sum (\( H_n \sim \log n \)).

Hints:

- Assume the proposals are made one at a time, in some arbitrary order. As each men makes a proposal, he proposes to a uniformly chosen women which he has not proposed to previously.
- Let \( Y_t \) be the number of women with no pending proposal after \( t \) proposals have been made. Show that \( Y_t \in \{ Y_{t-1}, Y_{t-1} - 1 \} \), and give a bound for the probability that \( Y_t = Y_{t-1} \).
- Show that the expected number of steps for \( Y_t \) to decrease from \( k \) to \( k - 1 \) is at most \( n/k \).