NOTE: Cooperative/coalition games were not covered this year.

20 marks 1. The game of Wythoff NIM is played as follows. There are two piles of tokens. A valid move is to take some number of tokens from one of the piles, or to take the same number from both. From position \((x, y)\) one can move to \((x - i, y)\) or \((x, y - i)\) or \((x - i, y - i)\), with \(i \geq 1\). You cannot leave a negative number of tokens anywhere. For example, from position \((2, 4)\) the valid moves are to \((1, 4)\), \((0, 4)\), \((2, 3)\), \((2, 2)\), \((2, 1)\), \((2, 0)\), \((1, 3)\), \((0, 2)\).

(a) Compute the Sprague-Grundy value of positions \((x, y)\) with \(x, y \leq 4\). (writing them in a grid helps.)

(b) What are the P-positions with \(x, y \leq 4\)?

(c) Prove that there are infinitely many P-positions.

(d) Alice and Bob play the sum of three copies of Wythoff NIM, with initial positions \((3, 0)\), \((3, 2)\), and \((3, 4)\). Alice moves first. What are the winning moves, if any?

14 marks 2. Consider the following coalition game with 3 players, Rose, Ray and Lou, who sell gloves. Rose and Ray have right gloves for sale, and Lou has a matching left glove. A tourist lost his right glove and is willing to pay $5 for a single right glove, and $20 for a matching pair of gloves (one for each hand). For a set of the players \(S\), let \(v(S)\) be the total profit achievable by \(S\).

(a) Write out the value \(v(S)\) for each subset \(S \subset \{Rose, Ray, Lou\}\)?

(b) What allocations are in the core?

(c) What is the Shapley-value allocation?

13 marks 3. (a) Find the value and (some) optimal strategies for the following 0-sum game.

\[
\begin{pmatrix}
6 & 0 & 2 & 7 \\
0 & 4 & 3 & 5 \\
1 & 5 & 5 & 3 \\
1 & 1 & 1 & 1
\end{pmatrix}
\]

(b) For what values of \(t\) does the game below have a saddle point?

(c) Find its value as a function of \(t\) for all \(t \in \mathbb{R}\).

\[
\begin{pmatrix}
0 & t \\
2t & 10
\end{pmatrix}
\]

20 marks 4. Consider the following game. Several people wish to build isolated houses along a road of length 1 Km (identified with the interval \([0,1]\)). The payoff for a player is the distance from their house to the nearest neighbour.

(a) If there are only two people, find a pure (non-random strategy) Nash equilibrium and a mixed Nash equilibrium.

(b) If there are 3 people (Alice, Bob, Carol), show that the only pure Nash equilibria are to have the three houses at \(\{0, 1/2, 1\}\) in some order.

(c) Show that if each of Alice, Bob, Carol picks uniformly one of the locations \(\{0, 1/2, 1\}\) this is not a Nash equilibrium.

(d) Find a mixed (not pure-strategy) Nash equilibrium.
(e) If Alice and Bob pick uniform locations on \([0, 1]\) for their houses, what is the optimal response for Carol?

5. Consider the general sum game with bi-matrix

\[
\begin{pmatrix}
(0, 6) & (4, 1) & (4, 4) \\
(3, 3) & (1, 4) & (0, 8) \\
(1, 1) & (1, 2) & (1, 3)
\end{pmatrix}
\]

(a) Find the safety levels for the two players.

(b) Find all Nash equilibria. (Justify your claims.)

6. Consider a cooperative game with bi-matrix form

\[
\begin{pmatrix}
(2, 4) & (4, 1) & (0, 0) \\
(5, 2) & (0, 5) & (0, 0) \\
(0, 0) & (0, 0) & (0, 0)
\end{pmatrix}
\]

(a) Find the transferrable utility (TU) solution.

(b) Find the Nash bargaining solution with fixed disagreement point \((0,0)\).

(c) Find the NTU solution and equilibrium exchange rate \(\lambda\).