1. (a) Define carefully what it means for events $A, B, C$ to be mutually independent.
(b) Define: $F$ is the c.d.f. of a random variable $X$.
(c) State Bayes’ formula.

Solution:
(a) For every subset, the probability of the intersection is the product of the probabilities.
(b) $F(x) = P(X \leq x)$.
(c) For a partition $(B_i)$ and event $A$,

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{\sum_j P(B_j)P(A|B_j)}.$$  

2. Suppose events $A, B$ satisfy $P(A) = 0.6$ and $P(B) = 0.4$ and $P(B|A) = 0.5$.
(a) Find $P(AB)$.
(b) Find $P(A \cup B)$.

Solution:
(a) We have $P(AB) = P(A)P(B|A) = 0.6 \cdot 0.5 = 0.3$.
(b) $P(A \cup B) = P(A) + P(B) - P(AB) = 0.6 + 0.4 - 0.3 = 0.7$.

3. Suppose events $A, B, C$ are independent and have $P(A) = 1/3$, $P(B) = 1/4$, $P(C) = 1/2$. Find the probability that exactly two of the three occurs.

Solution: The event is a disjoint union, and has probability

$$P(E) = P(ABC^c \cup AB^cC \cup A^cBC) = P(ABC^c) + P(AB^cC) + P(A^cBC)$$

$$= \frac{1}{3} \cdot \frac{3}{4} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{3}{4} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{5}{24}.$$  

4. You have two decks of cards: deck A has 16 cards containing numbers 1 to 4 in each of the four suits; Deck B has 21 cards containing numbers 1 to 7 in each of three suits. You select a deck at random and then draw two cards from that deck.

(a) What is the probability of getting a pair (matching numbers)?
(b) If you get a pair, what is the probability that it came from deck A?

**Solution:**
(a) Let $A, B$ be the event that you take deck $A$ or $B$, and $E$ the event that you draw a pair. So $P(A) = P(B) = 1/2$, and

$$P(E|A) = \frac{4(\binom{4}{2})}{(\binom{10}{2})} = 4 \cdot 616 \cdot 15/2 = \frac{1}{5} \quad P(E|B) = \frac{7(\binom{3}{2})}{(\binom{21}{2})} = \frac{7 \cdot 3}{21 \cdot 20/2} = \frac{1}{10}.$$  

Therefore

$$P(E) = P(A)P(E|A) + P(B)P(E|B) = \frac{P(E|A) + P(E|B)}{2} = \frac{3}{20}.$$  

(b) By Bayes’ theorem, this is

$$P(A|E) = \frac{P(A)P(E|A)}{P(A)P(E|A) + P(B)P(E|B)} = \frac{P(E|A)}{P(E|A) + P(E|B)} = \frac{2}{5}.$$  

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5. A circle of radius 1 is inscribed in the square $[-1, 1] \times [-1, 1]$. A point is chosen uniformly in the square. What is the probability that it lies outside the circle?

**Solution:** The area of the square is 4 and of the circle $\pi$. The complement is $4 - \pi$ and the probability is $\frac{4-\pi}{4}$.

6. Let $X$ be a random variable with p.m.f. given by $P(X = k) = c|k|$ for $k = -5, \ldots, 5$. What is $c$?

**Solution:** Since $\sum p(k) = 1$, we have $\sum_{k=-5}^{5} c|k| = 1$, but this sum is $30c$, so $c = \frac{1}{30}$.

7. Let $X$ be a random variable with p.m.f. given by $P(X = k) = \frac{c}{k}$ for $k = 10, \ldots, 30$. Compute $EX$ in terms of $c$.

**Solution:** We have $E[X] = \sum kp(k) = \sum_{k=10}^{30} k \cdot \frac{c}{k} = 21c$. 

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Let random variable $X$ have p.d.f.

$$F(x) = \begin{cases} cx^2 & 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $c$.
(b) Find the c.d.f. of $X$.
(c) Find $P(X > 2)$

**Solution:**

(a) Since $\int f(x) \, dx = 1$, we get

$$1 = \int_1^3 cx^2 \, dx = \left. \frac{cx^3}{3} \right|_1^3 = \frac{26}{3} c.$$ 

Thus $c = \frac{3}{26}$.

(b) The c.d.f. is

$$F(x) = \int_{-\infty}^{x} f(t) \, dt = \begin{cases} 0 & x < 1, \\ \frac{x^3}{26} & 1 \leq x \leq 3, \\ 1 & x \geq 3. \end{cases}$$