Gambler's Ruin:

If Heads $\rightarrow$ you win $1$, bank loses $1$
If Tails $\rightarrow$ other way around

$N$ in total, $n$ starting amount

Sol: We view the game as a two-step experiment:

stage 1: first flip
stage 2: remaining flips until game over
Now use law of total prob, conditioning on the outcome of stage 1:

\[ F = \text{"first flip is Heads"} \]

\[ E_n = \text{"you win, starting from $n"} \quad (0 \leq n \leq N) \]

\[ P(E_n) = P(E_n|F) \cdot P(F) + P(E_n|F^c) \cdot P(F^c) \]

\[ \overset{\uparrow}{P(E_{n+1})} \quad 1/2 \quad \overset{\uparrow}{P(E_{n-1})} \quad 1/2 \]

using law of total prob, with partition $\{F, F^c\}$.
Denote \( P_n = \mathbb{P}(E_n) \).

Then \[ P_n = \frac{1}{2} (P_{n+1} + P_{n-1}) \quad \text{(*)} \quad (1 \leq n \leq N-1) \]

Base cases: \( P_0 = 0 \), \( P_N = 1 \).

Rewrite (*) \( \rightarrow \quad 2P_n = P_{n+1} + P_{n-1} \)
\( \rightarrow \quad P_{n+1} - P_n = P_n - P_{n-1} \)

Thus, \( p_1 - p_0 = p_2 - p_1 = p_3 - p_2 = \ldots = p_N - p_{N-1} = d \)

\( \Rightarrow \quad Nd = (p_N - p_{N-1}) + (p_{N-1} - p_{N-2}) + \ldots + (p_2 - p_1) + (p_1 - p_0) \)
\( = P_N - P_0 = 1 \quad \Rightarrow \quad d = \frac{1}{N} \)
\[ p_1 = p_0 + d = 0 + d = d \]
\[ p_2 = p_1 + d = d + d = 2d \]
\[ p_3 = p_2 + d = 2d + d = 3d \]

\[ P_n = d \cdot n \]
Recall: (a) a random variable is a function \( X : S \rightarrow \mathbb{R} \).

(b) the event \( \{X \in A\} \) represents the event \( \{s \in S : X(s) \in A\} \).

(c) Binomial r.v.: \( X \sim \text{Bin}(n,p) \) counts successes among \( n \) independent trials of success prob. \( p \).

(d) Geometric r.v.: \( Y \sim \text{Geom}(p) \) measures number of attempts until first success in indep. trials.
Bernoulli r.v. with parameter $p \in (0,1)$.

Records whether a given event of prob. $p$ occurs or not.

Let $E$ be an event, $P(E) = p$.

Let $X = \mathbb{I}_E = \begin{cases} 1 & \text{if } E \text{ occurs} \\ 0 & \text{if } E \text{ does not occur} \end{cases}$. 

\[ X \sim \text{Ber}(p) \]

\[ \text{Bin}(1, p) \]

\[ \Pr(X=1) = p \]

\[ \Pr(X=0) = 1-p \]

$X \in \{0,1\}$, p.m.f.

indicator of $E$.
Negative Binomial r.v. with parameters \( r \in \mathbb{N} \) and \( p \in (0,1) \).

Let \( A_1, A_2, \ldots \) be independent events, each having prob. \( \Pr(A_i) = p \).

Let \( X = \# \) of trials until \( r \)-th success = \( r \)-th smallest \( i \) such that \( A_i \) occurs.

[when \( r=1 \), this is a geometric r.v.]

\( X \in \{ r, r+1, r+2, \ldots \} \)

p.m.f. \( \Pr(X=k) = \binom{k-1}{r-1} \cdot p^r \cdot (1-p)^{k-r} \)
Example: You play ping-pong against a friend who is a better player, winning 60% of each serve.

You were lucky and the current score is: 18 - 15

What is the prob. that you will win the game?
(you reach 21 before your friend does)

Sol: You need another 3 points.
Let \( X \) be the \# of serves when you first get to 21.

You win if by the time that happens, your friend has at most 20 points.
\[ X \sim \text{Neg Bin} \left(3, 0.4\right) \]

You win \( \iff \) \( 12 + X \leq 20 \iff X \leq 8 \)

After \( X \) serves: you have 21 points
your friend has \( 15 + (X - 3) \)

\[ = 12 + X \]

\[ P(\text{you win}) = P(X \leq 8) = \ldots. \]