Random Variables

\(X: \text{Time for first 1 in a sequence of coin tosses.}\)

\[ P(X = k) = \left(\frac{1}{2}\right)^k \quad \text{for} \quad k = 1, 2, 3, \ldots \]

Recall: A random variable is a function \(X\) defined on sample space.

Discrete R.V.: Possible values are some sequence \(x_1, x_2, \ldots\)

Continuous R.V.: Continuum.

E.g. \(S = \) a disc. Uniform probability.

Let \(X\) be distance to centre.
\[ P(X = \frac{R}{2}) = 0 \]

\[ P(X \leq \frac{R}{2}) : \frac{\pi (\frac{R}{2})^2}{\pi R^2} = \frac{1}{4} \]

e.g. (Discrete): value in darts:

A contin. RV, has \( P(X = a) = 0 \) for every \( a \).

e.g. Let \( X \) be time to get 6 in a seq. of dice.

e.g. Urn has a red ball, \( b \) blue balls.

\[ X = \text{time to find first blue ball, when sampling with replacement}. \]

\[ P(X = k) = \] \( k = 1, 2, 3, \ldots \)
\[ P(X=1) = \frac{b}{a+b} \]
\[ P(X=2) = \frac{a}{a+b} \cdot \frac{b}{a+b} \]
\[ P(X=k) = \frac{a^{k-1} \cdot b}{(a+b)^k} = \left(\frac{a}{a+b}\right)^{k-1} \cdot \frac{b}{a+b} = (1-q)^{k-1} \cdot q \]

where \( q = \frac{b}{a+b} \)

**Note:** \( \sum_{k=1}^{\infty} (1-q)^{k-1} \cdot q = q + q(1-q) + q(1-q)^2 + \ldots \)
\[ = \frac{q}{1-(1-q)} = 1. \]

**Def.:** The geometric R.V., with parameter \( q \), denoted \( \text{Geom}(q) \) has \[ P(X=k) = q \cdot (1-q)^{k-1} \]
for \( k=1, 2, 3, \ldots \)

The probability mass function (pmf) of a R.V. is the probability of each value. \( \text{Pa} = P(X=a) \)
Note: pmf is used only for discrete R.V.

Note: \( \sum_{a} p_{a} = 1 \) \( \sum \) sum over all possible values

**Binomial R.V.:**

\[
1 = (p + (1-p))^{n} = \sum_{k=0}^{n} \binom{n}{k} p^{k} (1-p)^{n-k}
\]

The **Binomial R.V.** \( \text{Bin}(n, p) \) has \( P(X=k) = \binom{n}{k} p^{k} (1-p)^{n-k} \)

e.g. from the same urn, sample \( n \) balls.

Let \( X = \# \text{ blue ball observed} \).

\[
P(X=k) = \frac{\binom{n}{k} b^{k} a^{n-k}}{(a+b)^{n}} = \binom{n}{k} \left( \frac{b}{a+b} \right)^{k} \left( \frac{a}{a+b} \right)^{n-k} = \binom{n}{k} q^{k} (1-q)^{n-k}
\]

with \( q = \frac{b}{a+b} \). \( X \) is \( \text{Bin}(n, q) \).
Conditional probability

Idea: Probability of E conditioned on F, if we know F happens, what is Probab. of E?

![Diagram of events F and E, with F shaded and E intersected with F]

Definition: $P(E|F) = \frac{P(E \cap F)}{P(F)}$ when $P(F) \neq 0$.

Prob. of E conditioned on F.
e.g. toss two dice.

\[ P(E) = \frac{11}{36} \quad P(F) = \frac{4}{36} \quad P(EF) = \frac{2}{36} \]

\[ P(E|F) = \frac{\frac{2}{36}}{\frac{4}{36}} = \frac{2}{4} = \frac{1}{2} \]

\[ F = \{ \text{sum is 5} \} = \{ (1,4), (2,3) \} \]

\[ E = \{ \text{one of the dice is 3} \} \]

\[ P(E|F) \text{ is same} \]