Math 302, assignment 8 solutions

1. A dartboard centered at the origin has radius 20. Let $(X, Y)$ be the random location of a dart thrown by a competent player. Assume $X$ and $Y$ have joint probability density function

$$f(x, y) = \begin{cases} c(20 - \sqrt{x^2 + y^2}) & \text{if } \sqrt{x^2 + y^2} \leq 20, \\ 0 & \text{if } \sqrt{x^2 + y^2} > 20. \end{cases}$$

(a) Find $c$.

(b) The bullseye is the center circle of radius 1. Find the probability that the player hits the bullseye.

Solution.
(a) Since the overall integral of the density is 1, we must have

$$1/c = \iint (20 - \sqrt{x^2 + y^2}) \, dx \, dy$$

where the integral is over the circle of radius 20. By using polar coordinates we find

$$1/c = \int_0^{20} \int_0^{2\pi} (20 - r) r \, d\theta \, dr = \frac{4000\pi}{3},$$

so $c = \frac{3}{4000\pi}$.

(b) This probability is the integral over the small circle of the p.d.f., which comes to

$$c \int_0^1 \int_0^{2\pi} (20 - r) r \, d\theta \, dr = \frac{3}{4000\pi} \frac{58\pi}{3} = \frac{58}{4000}.$$

2. Suppose $X$ is uniform in $[0, 1]$ and $Y$ is $N(0, 1)$, and they are independent. Find $P(X < Y)$.

Solution. In general, the probability that $X < Y$ is $\iint_{x<y} f(x, y) \, dx \, dy$. In our case, this is

$$\int_0^1 \int_0^y \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dx \, dy + \int_0^1 \int_y^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dx \, dy = \int_0^1 \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy + \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy$$

$$= \frac{-1}{\sqrt{2\pi}} e^{-y^2/2} \bigg|_0^1 + (1 - \Phi(1))$$

$$= \frac{1 - e^{-1/2}}{\sqrt{2\pi}} + (1 - \Phi(1)).$$

Integrating first over $y$ also works, using integration by parts.

3. If $X, Y$ have joint density $xe^{-x(1+y)}$ for $x, y > 0$ and 0 otherwise, find the marginal distributions of $X$ and $Y$. Are $X, Y$ independent?

Solution. Integrating over $y$ we get for $x > 0$:

$$f_X(x) = \int_0^\infty xe^{-x(1+y)} \, dy = e^{-x(1+y)} \bigg|_0^\infty = e^{-x}.$$

Therefore $X$ is an $\text{Exp}(1)$ random variable.
Integrating over \( x \) we get

\[
    f_Y(y) = \int_0^\infty xe^{-x(1+y)}dx = \frac{-x}{1+y}e^{-x(1+y)}\bigg|_0^\infty + \int_0^\infty \frac{1}{1+y}e^{-x(1+y)} = \frac{1}{(1+y)^2}.
\]

(This is the density for \( y > 0 \).

Since \( f(x, y) \neq f_X(x)f_Y(y) \), they are not independent.

4. If \( X, Y \) are independent \( \text{Exp}(\lambda_1) \) and \( \text{Exp}(\lambda_2) \), find the following:

   (a) \( P(X < Y) \).

   (b) distribution of \( \min(X, Y) \). (Hint: think of the c.d.f.)

   (c) distribution of \( X + Y \). (same hint.)

**Solution.**

(a) \( \min(X, Y) > t \) means \( X > t \) and \( Y > t \). These events are independent, so

\[
    P(\min(X, Y) > t) = P(X > t)P(Y > t) = e^{-\lambda_1 t}e^{-\lambda_2 t}.
\]

This shows that \( \min(X, Y) \) is exponential with parameter \( \lambda_1 + \lambda_2 \).

Alternative solution is to write \( P(\min(X, Y) \leq t) \) as a double integral, and compute that.

(b) Using the convolution formula from class, for \( t > 0 \), the density of \( X + Y \) at \( t \) is \( f_{X+Y}(t) = \int_0^t \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 (t-x)}dx \).

This evaluates to

\[
    f_{X+Y}(t) = \lambda_1 \lambda_2 \frac{e^{-\lambda_2 t} - e^{-\lambda_1 t}}{\lambda_1 - \lambda_2}.
\]

When \( \lambda_1 = \lambda_2 \) this becomes

\[
    \int_0^t \lambda \lambda_2 e^{-\lambda_2 t}dx = t\lambda_2 e^{-\lambda_1 t}.
\]

5. Assume \( X, Y \) are independent with

\[
    E(X) = 1 \quad E(Y) = 2 \quad \text{Var}(X) = 2^2 \quad \text{Var}(Y) = 3^2.
\]

Let \( U = 2X + Y \) and \( V = 2X - Y \).

   (a) Find \( E(U) \) and \( E(V) \).

   (b) Find \( \text{Var}(U) \) and \( \text{Var}(V) \).

   (c) Find \( \text{Cov}(U, V) \).

**Solution.**

(a) By linearity of expectation, \( E(2X + Y) = 2E(X) + E(Y) = 4 \) and \( E(2X - Y) = 2E(X) - E(Y) = 0 \).

(b) Since variance of sums of independent variables are independent,

\[
    \text{Var}(2X \pm Y) = 2^2 \text{Var}(X) + (-1)^2 \text{Var}(Y) = 16 + 9 = 25
\]

   and both have variance 25.

(c) From (a) we have

\[
    \text{Cov}(U, V) = E(UV) - E(U)E(V) = E(UV).
\]

This is \( E(2X + Y)(2X - Y) = E(4X^2 - Y^2) \). We have \( E(X^2) = \text{Var}(X) + (E(X))^2 = 5 \) and similarly \( E(Y^2) = 13 \), and so \( \text{Cov}(U, V) = 4 \cdot 5 - 13 = 7 \).