Math 302, assignment 6 solutions

1. Let $X$ take values in \{1, 2, 3, 4, 5\} and have p.m.f. given by

<table>
<thead>
<tr>
<th>$k$</th>
<th>$P(X = k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/7</td>
</tr>
<tr>
<td>2</td>
<td>1/14</td>
</tr>
<tr>
<td>3</td>
<td>3/14</td>
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<tr>
<td>4</td>
<td>2/7</td>
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<tr>
<td>5</td>
<td>2/7</td>
</tr>
</tbody>
</table>

(a) Calculate $P(X \leq 3)$

(b) Calculate $P(X < 3)$

(c) Calculate $P(X < 4.12 \mid X > 1.6)$

(d) Calculate $E[X]$  

(e) Calculate $E[X - 2]$  

(f) Calculate $Var(X)$

Solution.

(a) $P(X \leq 3) = 1/7 + 1/14 + 3/14 = 3/7$.

(b) $P(X < 3) = 1/7 + 1/14 = 3/14$.

(c) $P(X < 4.12 \mid X > 1.6) = \frac{P(1.6 < X < 4.12)}{P(X > 1.6)} = \frac{\frac{3}{14}}{\frac{2}{7} + \frac{5}{2}} = \frac{2}{3}$.

(d) $E(X) = 1 \cdot (1/7) + 2 \cdot (1/14) + 3 \cdot (3/14) + 4 \cdot (2/7) + 5 \cdot (2/7) = \frac{49}{14} = 3.5$.


(f) $E[X^2] = 1^2 \cdot (1/7) + 2^2 \cdot (1/14) + 3^2 \cdot (3/14) + 4^2 \cdot (2/7) + 5^2 \cdot (2/7) = \frac{197}{14}$. Therefore, $Var(X) = E[X^2] - (E[X])^2 = \frac{197}{14} - (3.5)^2 = \frac{51}{28}$.

2. A random variable has p.d.f.

$$f(x) = \begin{cases} e^x & 0 \leq x \leq a, \\ 0 & \text{otherwise,} \end{cases}$$

where $a > 0$ is some number.

(a) Find the value of $a$.

(b) Compute $E[X]$.

(c) Compute $Var(X)$.

Solution.

(a) Since $1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^a e^x dx = e^a - 1$, we deduce that $a = \ln 2$.

(b) $E[X] = \int_{-\infty}^{\infty} x f(x)dx = \int_0^{\ln 2} x e^x dx = \int_0^{\ln 2} (xe^x)' - e^x dx = 2 \ln 2 - 1$.

(c) We first find $E[X^2] = \int_0^{\ln 2} x^2 e^x dx = 2(\ln 2)^2 - 2E[X]$. Thus,

$$Var(X) = E[X^2] - (E[X])^2 = 2(\ln 2)^2 - 2(2 \ln 2 - 1) - (2 \ln 2 - 1)^2 = 1 - 2(\ln 2)^2.$$

3. An urn contains 5 black and 5 white balls. You draw balls one by one without replacement, until you draw your first white ball. The total number of balls you have drawn, including the first white ball, is a random variable $X$.
(a) Calculate the p.m.f. of $X$.
(b) Calculate the expectation $E X$.
(c) Calculate the variance $\text{Var}(X)$.

**Solution.**

(a) The values that $X$ can take are \{1, 2, 3, 4, 5, 6\}, and
$$
P(X = k) = \frac{\binom{10}{k-1}}{10^6} = \frac{5 (10-k)}{5 (10-k-1)}
$$

(b) We have
$$
E X = \sum_{k=1}^{6} k \binom{10}{k-1} = \frac{11}{6}
$$

(c) We have
$$
\sigma^2(X) = \sum_{k=1}^{6} (k - \frac{11}{6})^2 \binom{10}{k-1} = \frac{275}{252}
$$

4. Let $X$ be a Bin($n, p$) random variable. Show that $\text{Var}(X) = np(1-p)$.

Hint: First compute $E[X(X - 1)]$, similarly to the computation of $E[X]$ from class, and then use properties of expectation.

**Solution.** By the definition of expectation,
$$
E[X(X - 1)] = \sum_{k=0}^{n} k(k-1)P(X = k) = \sum_{k=2}^{n} k(k-1)P(X = k).
$$

Plugging in the p.m.f. of a Binomial r.v., we get
$$
E[X(X - 1)] = \sum_{k=2}^{n} k(k-1) \binom{n}{k} p^k (1-p)^{n-k}.
$$

Expanding the binomial coefficient, we get
$$
E[X(X - 1)] = n(n-1) p^2 \sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!(n-k)!} p^{k-2} (1-p)^{n-k}.
$$

Letting $\ell = k - 2$, we have
$$
E[X(X - 1)] = n(n-1) p^2 \sum_{\ell=0}^{n-2} \binom{n-2}{\ell} p^{\ell} (1-p)^{n-2-\ell}.
$$

The binomial theorem implies that the sum is 1 so that
$$
E[X(X - 1)] = n(n-1) p^2.
$$

Thus, by (d) of question 2, we have
$$
E X^2 = E[X^2 - X] + EX = E[X(X - 1)] + EX = n(n-1) p^2 + np.
$$

Finally,
$$
\text{Var}(X) = E X^2 - (EX)^2 = n(n-1) p^2 + np - (np)^2 = np(1-p).
$$

5. Let $X$ be a random variable (discrete or continuous). Define a function $f : \mathbb{R} \to \mathbb{R}$ by
$$
f(t) = E(X - t)^2.
$$

Show that $f$ attains its minimum at $t = EX$. 

Solution. By the linearity of expectation, we have
\[ f(t) = E[X^2 - 2tX + t^2] = EX^2 - 2tEX + t^2. \]
Thus, \( f \) is a quadratic function, and it is standard that its minimum is at \( t = EX \).

6. A fair coin is tossed 50 times. The outcomes are written in order, producing a 50-letter “word” consisting of the letters H and T. Compute the expected number of occurrences of “HHH” in this word (overlaps are allowed). For example, the word THHHHTTTHHTH has 3 such occurrences.

Solution. Let \( W_1, \ldots, W_{50} \) denote the outcomes of the coin tosses. For \( 1 \leq i \leq 48 \), let \( E_i \) denote the event that \( W_i = W_{i+1} = W_{i+2} = 'H' \). Let \( X_i \) denote the indicator of the event \( E_i \), that is, \( X_i = 1 \) if \( E_i \) occurs, otherwise \( X_i = 0 \). Then the number of occurrences of HHH is precisely \( X_1 + \cdots + X_{48} \).

Note that \( X_1, \ldots, X_{48} \) are not independent (!) so \( X_1 + \cdots + X_{48} \) is not a Bin(48, 1/8) random variable.

However, since each \( X_i \) is a Ber(1/8) r.v., using (d) of question 3, we conclude that \( E[X_1 + \cdots + X_{48}] = EX_1 + \cdots + EX_{48} = 48/8 = 6 \).

bonus In a city, there are on average 2.3 children in a family. A randomly chosen child has on average 1.6 siblings. Determine the variance of the number of children in a randomly chosen family.

Solution. Let \( X \) be the number of children in a randomly chosen family and let \( Y \) be the number of siblings of a randomly chosen child. Let \( n \) be the maximal number of children in a family, and assume that there are \( a_i \) families with exactly \( i \) children for each \( i = 0, 1, \ldots, n \). Then the total number of families is \( F = \sum_{i=0}^{n} a_i \) and the total number of children is \( C = \sum_{i=0}^{n} ia_i \). Thus we have
\[
P(X = i) = \frac{a_i}{F} \quad \text{and} \quad P(Y = i - 1) = \frac{ia_i}{C}
\]
for all \( i = 0, \ldots, n \). The definition of mean and our condition give
\[
E(X) = \sum_{i=0}^{n} iP(X = i) = \sum_{i=1}^{n} \frac{ia_i}{F} = \frac{C}{F} = 2.3, \tag{1}
\]
and similarly
\[
E(Y) = \sum_{i=0}^{n} (i-1)P(Y = i - 1) = -1 + \sum_{i=0}^{n} iP(Y = i - 1) = -1 + \sum_{i=0}^{n} \frac{i^2a_i}{C} = 1.6,
\]
so
\[
\sum_{i=0}^{n} \frac{i^2a_i}{C} = 2.6. \tag{2}
\]
Using (1) and (2) the second moment of \( X \) is
\[
E(X^2) = \sum_{i=0}^{n} i^2P(X = i) = \sum_{i=0}^{n} \frac{i^2a_i}{F} = \frac{C}{F} \sum_{i=0}^{n} \frac{i^2a_i}{C} = 2.3 \cdot 2.6.
\]
Thus
\[
\text{Var}(X) = E(X^2) - (EX)^2 = 2.3 \cdot 2.6 - 2.3^2 = 2.3 \cdot 0.3 = 0.69.
\]

Extra practice problems (do not hand in):
Chapter 3: 16,17,18,19,23,26,27