Math 302, assignment 5 solutions

1. Let $X$ be a r.v. taking values in $\{1, \ldots, 6\}$ with p.m.f. of the form

$$P(X = k) = ck.$$ 

(a) Find $c$.
(b) Find the probability that $X$ is odd.
(c) Find the c.d.f. of $X$.

**Solution.**

(a) Since $\sum_{i=1}^{6} p(i) = 1$, we find $c = \frac{1}{21}$.

(b) This is $P(X = 1) + P(X = 3) + P(X = 5) = c(1 + 3 + 5) = \frac{9}{21}$.

(c) One way to write this:

$$F(t) = \begin{cases} 0 & t < 1, \\ 1/21 & 1 \leq t < 2, \\ 3/21 & 2 \leq t < 3, \\ 6/21 & 3 \leq t < 4, \\ 10/21 & 4 \leq t < 5, \\ 15/21 & 5 \leq t < 6, \\ 1 & 6 \leq t. \end{cases}$$

For $0 \leq t < 7$ this can also be written as $F(t) = \lfloor t \rfloor (\lfloor t \rfloor - 1)/21$, where $\lfloor t \rfloor$ is the integer part of $t$.

2. Let $X$ be a r.v. taking values in $\{1, 2, \ldots\}$ with p.m.f. of the form

$$P(X = k) = \frac{c}{k(k+1)}.$$ 

(a) Show that $P(X \geq k) = c/k$ for any $k \geq 1$.
(b) Find $c$.
(c) * Find the probability that $X$ is odd.

**Solution.**

(a) $P(X \geq k) = \sum_{n=k}^{\infty} \frac{c}{n(n+1)}$. This is a telescoping sum (everything but the first term cancels out) giving the claim.

(b) Since $P(X \geq 1) = 1$ but this is $c/1$, we get $c = 1$.

(c) $P(X \text{ is odd}) = \sum_{n=1}^{\infty} \frac{c}{2n} - \frac{c}{2n+1} = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2$.

3. In a card game, 13 cards are given to you out of a deck of 52. This game is being played 1000 times. Identify (with names and parameters) the following random variables:

(a) The number of games in which all cards you receive have the same suit.
(b) The first time where the number of aces you receive is at least 1.
(c) The number of games in which you receive exactly three aces.
(d) The third time in which you received no aces.
Solution.
(a) In each game this occurs with probability \( p = \binom{39}{13}/\binom{52}{13} \). The number of such games is \( \text{Bin}(1000, p) \) with this \( p \).
(b) In each game you get no aces with probability \( q = \binom{48}{13}/\binom{52}{13} \). This is geometric \( \text{Geom}(1 - q) \).
(c) The probability of 3 aces is \( p = \binom{4}{3}\binom{48}{10}/\binom{52}{13} \). The number of such games is \( \text{Bin}(1000, p) \) with this \( p \).
(d) The is a negative Binomial: \( \text{NegBin}(3, \binom{48}{13}/\binom{52}{13}) \).

Note that we have cheated a little bit in b) and d): The geometric/negative binomial random variables would only be applicable if the game is played continuously (rather than “only” 1000 times). Since 1000 is a rather large number, it might still be a good approximation to use Geom/NegBin. The question was phrased ambiguously to not give the answers away.

4. Suppose that the continuous RV \( X \) has c.d.f. given by

\[
F(x) = \begin{cases} 
0 & \text{if } x < \frac{1}{\sqrt{2}} \\
5 - 12\sqrt{2}x + 18x^2 - 4\sqrt{2}x^3 & \text{if } \frac{1}{\sqrt{2}} \leq x < \sqrt{2} \\
1 & \text{if } \sqrt{2} \leq x
\end{cases}
\]

(a) Find the smallest interval \([a, b]\) such that \( P(a \leq X \leq b) = 1 \).
(b) Find \( P(0 < X < \frac{1}{2}) \).
(c) Find \( P(X = 1) \).
(d) Find \( P(1 < X < \frac{1}{2}) \).
(e) Find the p.d.f. of \( X \).

Solution.
(a) \( a = 1/\sqrt{2} \) and \( b = \sqrt{2} \).
(b) This is \( F(1/2) - F(0) = 0 - 0 \).
(c) \( P(X = a) = 0 \) for every \( a \).
(d) This is \( F(3/2) - F(1) = 1 - F(1) = -22 + 16\sqrt{2} \).
(e) \( f(x) = F'(x) = -12\sqrt{2} + 36x - 12\sqrt{2}x^2 \) on \([1/\sqrt{2}, \sqrt{2}]\) and 0 outside this interval.

5. Define the function \( f(x) = \begin{cases} 
9x^2 - 4x^3 + b & x \in [0, 1] \\
0 & \text{otherwise}
\end{cases} \). Show that there is no value of \( b \) for which this is the p.d.f. of some continuous RV.

Solution. We must have \( \int_{-\infty}^{\infty} f(x)dx = 1 \). This is \( \int_{0}^{1} 9x^2 - 4x^3 + bdx = 2 + b \), so \( b = -1 \). However, this means that \( f(x) < 0 \) for some values of \( x \) (any \( x \in [0, 1/3] \) for example.).

6. A stick of length \( \ell \) is broken into two pieces at a position \( X \sim \text{Unif}[0, \ell] \). Let \( Y \) denote the length of the smaller piece.
(a) Calculate the c.d.f. of \( Y \), that is, calculate \( P(Y \leq b) \).
(b) Calculate the p.d.f. of \( Y \). Can you identify what kind of random variable \( Y \) is?

Solution.
(a) The smaller segment can be anything from 0 to \( \ell/2 \). In order to get \( Y \leq b \) the uniform point \( X \) must be within \( \ell \) of either end of the stick, so \( F(b) = P(Y \leq b) = 2b/\ell \) for \( 0 \leq b \leq \ell \). It is 0 or 1 elsewhere.
(b) The pdf is \( F'(b) = \begin{cases} 
2/\ell & 0 \leq b \leq \ell/2, \\
0 & \text{otherwise}
\end{cases} \). This means \( Y \) is uniform on \([0, \ell/2]\).