1. We toss two dice. Consider the events

\[ E = \text{“The sum of the outcomes is odd”}, \]

\[ F = \text{“At least one outcome is 4”}. \]

Calculate the conditional probabilities \( P(E \mid F) \) and \( P(F \mid E) \).

**Solution.** \( P(E) = \frac{1}{2}, P(F) = \frac{11}{36}, P(E \cap F) = \frac{6}{36}, \) so

\[ P(E \mid F) = \frac{P(E \cap F)}{P(F)} = \frac{6/36}{11/36} = \frac{6}{11} \]

and

\[ P(F \mid E) = \frac{P(E \cap F)}{P(E)} = \frac{6/36}{1/2} = \frac{1}{3}. \]

2. Let \( X \) be a geometric random variable with a given parameter \( 0 < p < 1 \). Show that for all integers \( m, n \geq 1 \) we have

\[ P(X = n + m \mid X > n) = P(X = m) \]

and

\[ P(X > n + m \mid X > n) = P(X > m). \]

(For this reason, we say that a geometric random variable has no memory.)

**Solution.** We learned that \( P(X = m) = p(1 - p)^{m-1} \) and that \( P(X > m) = (1 - p)^m \), therefore by the definition of conditional probability we obtain

\[ P(X = n + m \mid X > n) = \frac{P(X = n + m)}{P(X > n)} = \frac{p(1 - p)^{n+m-1}}{(1 - p)^n} = p(1 - p)^{m-1} = P(X = m) \]

\[ P(X > n + m \mid X > n) = \frac{P(X > n + m)}{P(X > n)} = \frac{(1 - p)^{n+m}}{(1 - p)^n} = (1 - p)^m = P(X > m) \]

3. A fair die is rolled repeatedly.

(a) Give an expression for the probability that the first five rolls give a three at most two times.

(b) Calculate the probability that the first three does not appear before the fifth roll.

(c) Calculate the probability that the first three appears before the twentieth roll, but not before the fifth roll.

**Solution.**

a) Let \( X \) denote the number of threes on the first five rolls, then our probability is

\[ P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = \left(\frac{5}{6}\right)^5 + 5\left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right) + 10\left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right)^2. \]

b) The event in question is that none of the first four rolls is a three. On each die the probability of not rolling three is \( 5/6 \), so by independence the probability in question is \( (5/6)^4 \).

c) Let \( A \) be the event that none of the first four rolls is a three, and \( B \) be the event that some of the rolls from 5–19 is a three, then our event in question is \( A \cap B \). By part b) we have \( P(A) = (5/6)^4 \) and similarly we obtain \( P(B^c) = (5/6)^{15} \), so \( P(B) = 1 - (5/6)^{15} \). Since \( A \) and \( B \) are independent, we obtain

\[ P(A \cap B) = P(A)P(B) = (5/6)^4 \left(1 - (5/6)^{15}\right). \]
4. Let \( m \) be an integer chosen uniformly from \{1, \ldots, 100\}. Decide whether the following events are independent:
   (a) \( E = \{ m \text{ is odd} \} \) and \( F = \{ m \text{ is divisible by 7} \} \)
   (b) \( E = \{ m \text{ has two digits} \} \) and \( F = \{ m \text{ is divisible by 3} \} \)
   (c) \( E = \{ m \text{ is prime} \} \) and \( F = \{ \text{one of the digits of } m \text{ is a 4} \} \)

Solution.  
   a) \( P(E)P(F) = \frac{1}{2} \cdot \frac{14}{100} = \frac{7}{100} = P(E \cap F) \), so \( E \) and \( F \) are independent.

   b) \( P(E) = 91/100, P(F) = 33/100, \) and \( P(E \cap F) = 30/100. \) The events \( E \) and \( F \) are not independent since \( P(E \cap F) \neq P(E)P(F) \).

   c) \( P(E) = 25/100, P(F) = 19/100, \) and \( P(E \cap F) = 3/100. \) The events \( E \) and \( F \) are not independent since \( P(E \cap F) \neq P(E)P(F) \).

5. Draw 5 cards at random from a standard deck of 52 cards. Find
   (a) \( P(\text{full house} | \text{full-house or three-of-a-kind}) \)
   (b) \( P(\text{full house} | \text{at least two aces}) \)
   (c) \( P(\text{at least two aces} | \text{full-house}) \)

Solution.
   (a) The number of “full house” hands is \( \binom{13}{1} \binom{12}{1} \binom{4}{3} \). The number of “three-of-a-kind” hands is \( \binom{13}{1} \binom{4}{3} \binom{12}{1} \binom{4}{3}^2 \). The total number of poker hands is \( \binom{52}{5} \) (but anyway this will cancel out). Thus,

   \[
P(\text{full house} | \text{full-house or three-of-a-kind}) = \frac{P(\text{full house})}{P(\text{full-house or three-of-a-kind})} = \frac{\binom{13}{1} \binom{12}{1} \binom{4}{3}}{\binom{13}{1} \binom{4}{3} \binom{12}{1} \binom{4}{3}^2} = \frac{12 \cdot 6}{12 \cdot 6 + 66 \cdot 4^2} = \frac{3}{47}.
\]

(b) The number of hands with exactly two aces is \( \binom{4}{2} \binom{48}{3} \). Similarly, the number of hands with exactly three aces is \( \binom{4}{3} \binom{48}{2} \), and with exactly four is \( \binom{4}{4} \binom{48}{1} \). Also, the number of “full house” hands having at least two aces is \( \binom{4}{1} \binom{12}{1} \binom{4}{3} + \binom{4}{1} \binom{12}{1} \binom{4}{2} \). Thus,

   \[
P(\text{full house} | \text{at least two aces}) = \frac{P(\text{full house with at least two aces})}{P(\text{at least two aces})} = \frac{\binom{4}{2} \binom{12}{1} \binom{4}{3}}{\binom{4}{2} \binom{48}{3} + \binom{4}{3} \binom{48}{2} + \binom{4}{4} \binom{48}{1}} = \frac{576}{108336} = \frac{12}{2257}.
\]

(c) By the above, we have

   \[
P(\text{at least two aces} | \text{full house}) = \frac{P(\text{full house with at least two aces})}{P(\text{full house})} = \frac{2 \binom{4}{2} \binom{12}{1} \binom{4}{3}}{\binom{13}{1} \binom{4}{3} \binom{12}{1} \binom{4}{2}} = \frac{2}{13}.
\]

6. Three buses are stuck at a traffic jam. On first bus there are 24 men and 12 women, on the second 20 men and 15 women, and on the third 14 men and 21 women. A uniformly chosen bus is picked, and a uniformly random passenger from that bus is selected. What is the probability that the selected passenger is a man?
Solution. Let $E$ be the event that the selected passenger is a man. Let $F_1$, $F_2$ and $F_3$ be the events that the first, second and third bus are selected, respectively. By the law of total probability, we have

$$P(E) = P(E \mid F_1)P(F_1) + P(E \mid F_2)P(F_2) + P(E \mid F_3)P(F_3)$$

$$= \frac{24}{24 + 12} \cdot \frac{1}{3} + \frac{20}{20 + 15} \cdot \frac{1}{3} + \frac{14}{14 + 21} \cdot \frac{1}{3} = \frac{1}{3} \left( \frac{2}{3} + \frac{4}{7} + \frac{2}{5} \right).$$