1. Let $X$ and $Y$ be two independent uniform random variables on $(0, 1)$.
   (a) Using the convolution formula, find the p.d.f. of the random variable $Z = X + Y$, and graph it.
   (b) What is the moment generating function of $Z$?

2. Suppose that $X$ has moment generating function
   \[ M_X(t) = \frac{1}{3} + \frac{1}{2} e^{-t} + \frac{1}{6} e^{2t}. \]
   (a) Find the mean and variance of $X$ by differentiating the m.g.f. above.
   (b) Find the p.m.f. of $X$. Use your expression for the p.m.f. to check your answers from part (a).

3. Suppose that $X_1, \ldots, X_n$ are independent continuous random variables that all have the same c.d.f. $F(x)$. Define the random variable
   \[ Y = \max\{X_1, \ldots, X_n\}. \]
   Compute the c.d.f. and the p.d.f. of $Y$. Your answer should be in terms of $F(x)$. Hint: Express an inequality of the kind $\max\{X_1, \ldots, X_n\} \leq b$ in terms of separate inequalities for each $X_i$.

4. Let $X_n \sim \text{Bin}(n, \frac{\lambda}{n})$ for some $\lambda > 0$. Show that the moment generating function $M_{X_n}(t)$ converges as $n \to \infty$, and show that the limit is the m.g.f. of the Poisson($\lambda$) random variable.

5. You are given two coins that are optically indistinguishable, but one of them is fair, while the other will flip “Head” 51% of the time. To find out which is the fair one, you choose the following strategy:
   Pick one of the coins randomly, flip it $n$ times, and record $\bar{\mu}_n = \frac{1}{n} \times \text{(the number of heads flipped)}$. If $\bar{\mu}_n$ is closer to 50% than to 51%, you decide that the coin is the fair coin, otherwise, you decide that it is the biased coin.
   (a) Use Chebyshev’s inequality to find a value for $n$ such that you can be 90% sure that this procedure will identify the fair coin correctly.
   (b) Now use the central limit theorem and the Φ-table to find a better (smaller) value for $n$. How much smaller is your new $n$?

6. Suppose a random number generator generates 20 numbers per second, where each number is drawn uniformly from the interval $[9,10]$, independently of all other numbers. We are interested in the event that one of the drawn numbers is very close to Usain Bolt’s 100m sprint world record, that is, that this number belongs to the interval $[9.575, 9.585]$.
   (a) Suppose that the random number generator runs for 10 seconds. Use the Poisson approximation to estimate the probability that it produces more than 4 numbers that fall in that interval.
   (b) We now let the generator run indefinitely. Use the exponential random variable to estimate the probability that it produces the first such number before Usain Bolt finishes his run, i.e. within 9.58 seconds.
   (c) Suppose we let the random number generator compete against Usain Bolt, as described in (b), every day for 100 days. Use the normal random variable to approximate the probability that Usain Bolt wins at least 12 of these competitions.