Mathematics 101 — Solutions for week 9 quiz

Question 1 — 4 marks

Use Taylor series to compute $\displaystyle\int \frac{\sin(z)}{z}\,\mathrm{d}z$.

Question 2 — 6 marks

(a) Compute $\displaystyle\int_0^1 x^q\,\mathrm{d}x$ for $-1 < q < 0$.

(b) Compute $\displaystyle\int_0^1 x^q\,\mathrm{d}x$ for $q < -1$.

Solution 1

- The taylor series for $\sin(z)$ is

$$\sin(z) = \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$

- Hence the Taylor series for $\sin(z)/z$ is

$$\frac{\sin(z)}{z} = \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)!} z^{2n}$$

- Integrating this term by term gives

$$\begin{aligned}
\int \frac{\sin(z)}{z}\,\mathrm{d}z &= \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)!} \int z^{2n}\,\mathrm{d}z \\
&= c + \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)!} \cdot \frac{z^{2n+1}}{2n+1} \\
&= c + \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)!(2n+1)} z^{2n+1}
\end{aligned}$$

Solution 2a

- The function has a singularity at $x = 0$, so we need to use limits

$$\begin{aligned}
\int_0^1 x^q\,\mathrm{d}x &= \lim_{a\to 0^+} \int_a^1 x^q\,\mathrm{d}x \\
&= \lim_{a\to 0^+} \left[\frac{x^{q+1}}{q+1}\right]_a^1 = \lim_{a\to 0^+} \left[\frac{1}{q+1} - \frac{a^{q+1}}{q+1}\right] \\
&= \frac{1}{q+1} - \frac{1}{q+1} \cdot \lim_{a\to 0^+} a^{q+1}
\end{aligned}$$

- Now since $q > -1$, this limit is defined and is zero and so

$$\int_0^1 x^q\,\mathrm{d}x = \frac{1}{q+1}$$

Solution 2b

- We do the same thing

$$\int_0^1 x^q \, \mathrm{d}x = \frac{1}{q+1} - \frac{1}{q+1} \cdot \lim_{a \to 0^+} a^{q+1}$$

- Now $q < -1$ so $q + 1 < 0$ so the limit does not exist.
- Hence the integral is divergent.