Mathematics 101 — Solutions for week 6 quiz

Question 1 — 6 marks

Compute the following integrals. Remember that $\exp(x) = e^x$.

(a) $\displaystyle\int (\sin\theta)^3(\cos\theta)^2 \,\mathrm{d}\theta$

(b) $\displaystyle\int z\sin z \,\mathrm{d}z$

(c) $\displaystyle\int_0^{\pi/2} \frac{\cos y}{\sqrt{\sin y}} \cdot \exp\left(\sqrt{\sin y}\right) \,\mathrm{d}y$

Question 2 — 4 marks

Compute the following integral: $\displaystyle\int t\sec^2 t \,\mathrm{d}t$.

Solutions 1 — 2 marks each

(a) Hold on to a single power of $\sin\theta$ and change $\sin^2\theta$ into $1-\cos^2\theta$, and then sub $u = \cos\theta$:

$$\begin{aligned}
\int (\sin\theta)^3(\cos\theta)^2 \,\mathrm{d}\theta &= \int \sin^2\theta\cos^2\theta\sin\theta \,\mathrm{d}\theta \\
&= \int (1-\cos^2\theta)\cos^2\theta\sin\theta \,\mathrm{d}\theta \\
&= -\int (1-u^2)u^2 \,\mathrm{d}u = -\int (u^2-u^4) \,\mathrm{d}u \\
&= -u^3/3 + u^5/5 + C = -\frac{1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta + C
\end{aligned}$$

(b) Integration by parts. Choose $f = z$ and $g' = \sin z$. Hence $f' = 1$ and $g = -\cos z$

$$\begin{aligned}
\int fg' \,\mathrm{d}z &= fg - \int f'g \,\mathrm{d}z \\
\int z\sin z \,\mathrm{d}z &= -z\cos z + \int \cos z \,\mathrm{d}z \\
&= \sin z - z\cos z + c
\end{aligned}$$

(c) Substitution integral. Put $u = \sqrt{\sin y}$. So $\frac{\mathrm{d}u}{\mathrm{d}y} = \frac{\cos y}{2\sqrt{\sin y}}$

$$\begin{aligned}
\int_0^{\pi/2} \frac{\cos y}{\sqrt{\sin y}} \cdot \exp\left(\sqrt{\sin y}\right) \,\mathrm{d}y &= \int_{y=0}^{y=\pi/2} 2\frac{\mathrm{d}u}{\mathrm{d}y} e^u \,\mathrm{d}y \\
&= \int_{y=0}^{y=\pi/2} 2e^u \,\mathrm{d}u \\
&= [2e^u]_{y=0}^{y=\pi/2} = 2\left[\exp(\sqrt{\sin y})\right]_0^{\pi/2} \\
&= 2(e^1 - e^0) = 2e - 2
\end{aligned}$$

Solutions 2

Integrate by parts. Choose $f = t$ and $g' = \sec^2 t$, so $f' = 1$ and $g = \tan t$

$$\int fg' \, \mathrm{d}t = fg - \int f'g \, \mathrm{d}t$$
$$\int t \sec^2 t \, \mathrm{d}t = t \tan t - \int \tan t \, \mathrm{d}t$$

This integral is either from memory, or a substitution one:

$$\begin{aligned} \int \tan t \, \mathrm{d}t &= \int \frac{\sin t}{\cos t} \, \mathrm{d}t & \qquad u = \cos t, u' = -\sin t \\ &= -\int \frac{1}{u} \frac{\mathrm{d}u}{\mathrm{d}t} \, \mathrm{d}t = -\int 1/u \, \mathrm{d}u \\ &= -\log|u| + c = -\log|\cos t| + c \end{aligned}$$

Hence the integral we want is:

$$\int t \sec^2 t \, \mathrm{d}t = t \tan t + \log|\cos t| + c$$