Mathematics 101 — Solutions for week 4 quiz

Question 1 (3 marks)

Consider the region bounded by the curves $y = x^2$ and $y = qx$ where q is a positive real number. Find the area of this region as a function of q and determine when it is equal to 1.

Question 2 (7 marks)

Let R be the region enclosed by the curves $y = x^2 + 1$ and $y = 2x + 4$.

(a) Carefully plot R. **(1 marks)**

(b) Find the area of R. **(2 marks)**

(c) Find the volume of the solid obtained by rotating R about the line $y = -1$.

Do not completely evaluate the integral. It is sufficient to stop at an expression of the form "$F(b) - F(a)$" where F is an antiderivative of the integrand. **(4 marks)**

Solutions — 1

- The curves $y = x^2$ and $y = qx$ intersect at $x = 0$ and $x = q$.

$$x^2 = qx$$
$$x^2 - qx = x(x - q) = 0$$

- So the total area is

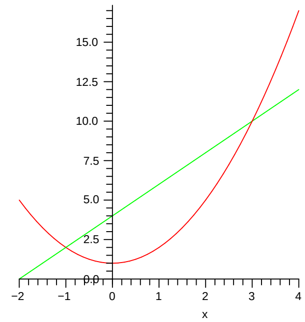
$$\begin{aligned} A &= \int_0^q (x^2 - qx)\,\mathrm{d}x \\ &= [x^3/3 - qx^2/2]_0^q \\ &= q^3/3 - q^3/2 = q^3/6 \end{aligned}$$

- So this area is one, when $q = 6^{1/3}$.

Solutions — 2

(a) The curve $y = x^2 + 1$ intercepts the y-axis at 1 and does not intercept the x-axis. The line $y = 2x + 4$ intercepts the y-axis at 4 and the x-axis at -2. The curves intersect each other at $x = -1, +3$

$$x^2 + 1 = 2x + 4$$
$$x^2 - 2x - 3 = (x - 3)(x + 1) = 0$$



(b) Since the line $y = 2x + 4$ lies above the quadratic, the total area is

$$\begin{aligned} A &= \int_{-1}^{3} ((2x+4) - (x^2+1))\, \mathrm{d}x = \int_{-1}^{3} (-x^2 + 2x + 3)\, \mathrm{d}x \\ &= \left[-x^3/3 + x^2 + 3x\right]_{-1}^{3} \\ &= (-27/3 + 9 + 9) - (1/3 + 1 - 3) \\ &= 9 - (-5/3) = 32/3 \end{aligned}$$

(c) The volume of revolution is given by the volume of the outer object - the volume of the inner object.

$$V = \pi \int_{-1}^{3} (f(x)^2 - g(x)^2)\, \mathrm{d}x$$

The outer curve is $2x + 4$ and the inner curve is $x^2 + 1$, but we are rotating around $y = -1$, so we need to increase the radii by 1:

$$\begin{aligned} V &= \pi \int_{-1}^{3} ((2x+5)^2 - (x^2+2)^2)\, \mathrm{d}x \\ &= \pi \int_{-1}^{3} (4x^2 + 20x + 25 - x^4 - 4x^2 - 4)\, \mathrm{d}x \\ &= \pi \int_{-1}^{3} (21 + 20x - x^4)\, \mathrm{d}x \\ &= \pi \left[21x + 10x^2 - x^5/5\right]_{-1}^{3} \\ &= \pi((63 + 90 - 243/5) - (-21 + 10 - 1/5)) \qquad \text{can stop here} \\ &= \frac{576}{5}\pi \end{aligned}$$