Mathematics 101 — Solutions for week 3 quiz

Question 1 (2 marks)

Let $g(t)$ be a continuous function, and define $f(x) = \int_1^{(e^x)} g(t)\,\mathrm{d}t$. What is $f'(x)$?

Question 2 (8 marks)

Evaluate the following integrals:

(a) $\int_1^3 \frac{1}{x^3}\,\mathrm{d}x$

(b) $\int \sqrt{z}(1-z^2)\,\mathrm{d}z$

(c) $\int_{-1}^1 (a - b\,|t|)\,\mathrm{d}t$

(d) $\int \sin(x)\sqrt{e^{\cos(x)}}\,\mathrm{d}x$

Solutions — each problem is 2 marks

Question 1

- We have $f(x) = \int_1^{(e^x)} g(t)\,\mathrm{d}t$.
- Let $u(x) = e^x$. Then define

$$h(u) = \int_1^u g(t)\,\mathrm{d}t$$

So that $f(x) = h(u(x))$.

- Now $\frac{\mathrm{d}f}{\mathrm{d}x}$ can be computed using the chain rule.

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}x} &= \frac{\mathrm{d}h}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \\ &= g(u)e^u \\ &= g(e^x)e^x\end{aligned}$$

Question 2

(a) Straightforward lookup.

$$\begin{aligned}\int_1^3 x^{-3}\,\mathrm{d}x &= \left[\frac{1}{-2}x^{-2}\right]_1^3 \\ &= -1/18 - (-1/2) = 8/18 = 4/9\end{aligned}$$

(b) Another straightforward lookup

$$\begin{aligned}\int \sqrt{z}(1-z^2)\,\mathrm{d}z &= \int (z^{1/2}-z^{5/2})\,\mathrm{d}z\\ &= \frac{1}{3/2}z^{3/2}-\frac{1}{7/2}z^{7/2}+c\\ &= \frac{2}{3}z^{3/2}-\frac{2}{7}z^{7/2}+c\end{aligned}$$

(c) Have to split the interval

$$\begin{aligned}\int_{-1}^{1}(a-b|t|)\,\mathrm{d}t &= \int_{-1}^{0}(a+bt)\,\mathrm{d}t+\int_{0}^{1}(a-bt)\,\mathrm{d}t\\ &= [at+bt^2/2]_{-1}^{0}+[at-bt^2/2]_{0}^{1}\\ &= 0-(-a+b/2)+(a-b/2)-0=2a-b\end{aligned}$$

(d) Substitution rule

$$\int \sin(x)\sqrt{e^{\cos(x)}}\,\mathrm{d}x = \int \sin(x)e^{\cos(x)/2}\,\mathrm{d}x$$

Put $u=\cos(x)/2$ and $\frac{\mathrm{d}u}{\mathrm{d}x}=-\sin(x)/2$ (or $\mathrm{d}x=\frac{-2}{\sin(x)}\,\mathrm{d}u$)

$$\begin{aligned}\int \sin(x)e^{\cos(x)/2}\,\mathrm{d}x &= \int -2e^{u}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x\\ &= -2\int e^{u}\,\mathrm{d}u\\ &= -2e^{u}+c=-2e^{\cos(x)/2}+c\end{aligned}$$