Mathematics 101 — Solutions for week 2 quiz

Question 1 = 10 marks

Consider the curve $y = f(x) = 6x^2 - 1$.

(a) Show that

$$5 \le \int_1^2 f(x)\mathrm{d}x \le 23.$$

3 marks

(b) Use Riemann sums to compute the signed area between this curve and the x-axis, between $x = 1$ and $x = 2$. Use $x_i^* = x_i$.

7 marks

Solution

Part (a)

- We bound the integral by bounding the function.
- Since $'f(x) = 12x$, the function is increasing on the interval $[1, 2]$.
- Hence its minimum value occurs at $x = 1$, namely $f(1) = 5$. Its maximum value occurs at $x = 2$, namely $f(2) = 24 - 1 = 23$.
- So we can bound the integral above and below by

$$\int_1^2 5\mathrm{d}x = 5 \le \int_1^2 f(x)\mathrm{d}x \le \int_1^2 23\mathrm{d}x = 23$$

Note — it was not sufficient to write "Since $f(1) = 5$ and $f(2) = 23$ the function is bounded between 5 and 23" — you need to show that the function does not have any turning points between $x = 1$ and $x = 2$.

Note — Since I did not specify how to bound the integral, I also accepted "$\int_1^2 (6x^2 - 1)\mathrm{d}x = [2x^3 - x]_1^2 = 13$. since the integral = 13, it is greater than 5 and less than 23". If you got part (b) correct then I would also accept "By my solution to part (b), the integral is 13, and so is bounded below by 5 and above by 23".

Part (b)

- Split the interval between 1 and 2 into n segments of equal width $= 1/n$.
- Let $x_i = 1 + i/n$ — so that the i^{th} segment is $[1 + (i - 1)/n, 1 + i/n]$.
- The height of the i^{th} rectangle is

$$f(x_i) = f(1 + i/n) = 6\left(1 + \frac{i}{n}\right)^2 - 1$$

so, its area is

$$f(x_i)\Delta x_i = \left(6\left(\frac{i}{n}\right)^2 - 12\frac{i}{n} + 6 - 1\right) \cdot \frac{1}{n}$$

- So the Riemann sum is

$$\begin{aligned} S_n &= \sum_{i=1}^{n} f(x_i)\Delta x_i \\ &= \frac{1}{n}\sum_{i=1}^{n}(6(i^2/n^2) + 12\frac{i}{n} + 5) \\ &= \frac{6}{n^3}\sum_{i=1}^{n} i^2 + \frac{12}{n^2}\sum_{i=1}^{n} i + \frac{5}{n}\sum_{i=1}^{n} 1 \\ &= \frac{6}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{12}{n^2}\frac{n(n+1)}{2} + 5 \end{aligned}$$

- So as $n \to \infty$ the Riemann sum limits to

$$\begin{aligned} \lim_{n\to\infty} S_n &= \lim_{n\to\infty}\frac{6}{6}\frac{n(n+1)(2n+1)}{n^3} + \lim_{n\to\infty}\frac{12}{2}\frac{n(n+1)}{n^2} + 5 \\ &= 2 + 6 + 5 = 13 \end{aligned}$$

- Hence the area under the curve is 13.