Mathematics 101 — Solutions to Second Midterm

Question 1 — 10 marks

Compute the following integrals

(a) $\displaystyle\int x \sin\left(\frac{x}{2}\right)\, \mathrm{d}x$

(b) $\displaystyle\int \tan\theta \, \sec^4\theta \, \mathrm{d}\theta$

Solutions — 5 marks each

(a) This is integration by parts.

- Put $f = x$ and $g' = \sin(x/2)$.
- Hence $f' = 1$ and $g = -2\cos(x/2)$
- Sub into the formula

$$\begin{aligned}
\int fg' \, \mathrm{d}x &= fg - \int f'g \, \mathrm{d}x \\
\int x\sin(x/2) \, \mathrm{d}x &= -2x\cos(x/2) + \int 2\cos(x/2) \, \mathrm{d}x \\
&= -2x\cos(x/2) + 4\sin(x/2) + c
\end{aligned}$$

(b) This is a trig integral.

- Try to make the substitution $u = \tan(\theta)$.
- Hence $\mathrm{d}u = \sec^2(\theta) \, \mathrm{d}\theta$
- So the integral becomes

$$\begin{aligned}
\int \sec^4\theta \tan\theta \, \mathrm{d}\theta &= \int \tan\theta \sec^2\theta \left(\sec^2\theta \, \mathrm{d}\theta\right) \\
&= \int \tan\theta \left(1 + \tan^2\theta\right) \left(\sec^2\theta \, \mathrm{d}\theta\right) \\
&= \int u(1+u^2) \, \mathrm{d}u \\
&= u^2/2 + u^4/4 + c \\
&= \frac{1}{2}\tan^2\theta + \frac{1}{4}\tan^4\theta + c
\end{aligned}$$

Question 2 — 10 marks

Compute the following integrals

(a) $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{1}{9-x^2}\,\mathrm{d}x$

Solutions — 5 marks each

(a) This is a trig substitution $x = 3\sin\theta$, so $\mathrm{d}x = 3\cos\theta\,\mathrm{d}\theta$.

$$\begin{aligned}\int \frac{1}{\sqrt{9-x^2}}\,\mathrm{d}x &= \int \frac{3\cos\theta\,\mathrm{d}\theta}{\sqrt{9-9\sin^2(\theta)}} \\ &= \int \frac{3\cos\theta\,\mathrm{d}\theta}{\sqrt{9\cos^2\theta}} \\ &= \int \frac{3\cos\theta\,\mathrm{d}\theta}{3\cos\theta} \\ &= \int 1\,\mathrm{d}\theta \\ &= \theta + c \\ &= \arcsin(\theta/3) + c\end{aligned}$$

(b) This is a partial fractions question

$$\begin{aligned}\frac{1}{9-x^2} &= \frac{1}{(3-x)(3+x)} \\ &= \frac{A}{3-x} + \frac{B}{3+x} \\ &= \frac{A(3+x)+B(3-x)}{(3-x)(3+x)} = \frac{x(A-B)+(3A+3B)}{(3-x)(3+x)}\end{aligned}$$

Matching coefficients of x gives

$$0 = A - B \qquad\qquad 1 = 3A + 3B$$

Hence $A = B = 1/6$. Now we can integrate

$$\begin{aligned}\int \frac{1}{9-x^2}\,\mathrm{d}x &= \int \left(\frac{1/6}{3-x} + \frac{1/6}{3+x}\right)\,\mathrm{d}x \\ &= -\frac{1}{6}\log|3-x| + \frac{1}{6}\log|3+x| + c \\ &= \frac{1}{6}\log\left|\frac{3+x}{3-x}\right| + c\end{aligned}$$

Question 3 — 5 marks

Let $y = \dfrac{2}{3}x^{\frac{3}{2}} + 7$.

(a) Find the length of the curve between $x = 0$ and $x = 2$.

Solutions — 5 marks

(a) We need to use the formula.

$$S = \int \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

Now — $f(x) = \frac{2}{3}x^{3/2} + 7$, so $f'(x) = x^{1/2}$.

$$\begin{aligned} S &= \int_0^2 \sqrt{1 + (x^{1/2})^2}\,\mathrm{d}x = \int_0^2 \sqrt{1+x}\,\mathrm{d}x \\ &= \left[\frac{2}{3}(1+x)^{3/2}\right]_0^2 \\ &= \frac{2}{3}3^{3/2} - \frac{2}{3} \\ &= 2\sqrt{3} - \frac{2}{3} \end{aligned}$$

Question 4 — 10 marks

Do the following integrals converge or diverge? If the integral converges then evaluate it.

(a) $\displaystyle\int_1^3 \frac{\mathrm{d}x}{x-1}$

(b) $\displaystyle\int_{\sqrt{3}}^\infty \frac{\mathrm{d}z}{1+z^2}$

Solutions — 4,6 marks

(a) The integrand has a singularity at $x = 1$, so we need to use limits

$$\begin{aligned}\int_1^3 \frac{\mathrm{d}x}{x-1} &= \lim_{a\to 1^+} \int_a^3 \frac{\mathrm{d}x}{x-1} \\ &= \lim_{a\to 1^+} \left[\log|x-1|\right]_a^3 \\ &= \lim_{a\to 1^+} \left(\log 2 - \log|a-1|\right)\end{aligned}$$

This limit is divergent. So the integral does not exist.

(b) The interval is infinite, so we use limits

$$\int_{\sqrt{3}}^\infty \frac{\mathrm{d}z}{1+z^2} = \lim_{b\to\infty} \int_{\sqrt{3}}^b \frac{\mathrm{d}z}{1+z^2}$$

Either recognise this as $\arctan(z)$ or... This is a trig substitution integral with $z = \tan\theta$. So $\mathrm{d}z = \sec^2\theta\,\mathrm{d}\theta$:

$$\begin{aligned}\int \frac{\mathrm{d}z}{1+z^2} &= \int \frac{\sec^2\theta\,\mathrm{d}\theta}{1+\tan^2\theta} \\ &= \int \frac{\sec^2\theta\,\mathrm{d}\theta}{\sec^2\theta} \\ &= \int 1\,\mathrm{d}\theta = \theta + c \\ &= \arctan^{-1} x + c\end{aligned}$$

Hence the integral is

$$\begin{aligned}\lim_{b\to\infty} \int_{\sqrt{3}}^b \frac{\mathrm{d}z}{1+z^2} &= \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1}\sqrt{3}\right) \\ &= \lim_{b\to\infty} \tan^{-1} b - \pi/3 \\ &= \pi/2 - \pi/3 = \pi/6\end{aligned}$$

Question 5 — 10 marks

The following function is a probability density function:

$$f(x) = \begin{cases} 0 & x < 0 \\ ax + b & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$

where a and b are some real numbers. The mean of this distribution is 1/3.

(a) Use this information to find a and b.

(b) Compute the median of this distribution.

Solutions — 5 marks each

(a) Since it is a PDF it must integrate to 1.

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f(x)\, \mathrm{d}x \\ &= \int_0^1 (ax + b)\, \mathrm{d}x \\ &= \left[ax^2/2 + bx\right]_0^1 \\ &= a/2 + b \end{aligned}$$

We also know its mean is 1/3, so

$$\begin{aligned} 1/3 &= \int_{-\infty}^{\infty} x f(x)\, \mathrm{d}x \\ &= \int_0^1 (ax^2 + bx)\, \mathrm{d}x \\ &= \left[ax^3/3 + bx^2/2\right]_0^1 \\ &= a/3 + b/2 \end{aligned}$$

This gives 2 equations to solve

$$a/2 + b = 1 \qquad\qquad a/3 + b/2 = 1/3$$

Hence $a + 2b = 2$ and $a + 3b/2 = 1$, so $b/2 = 1$ and $b = 2$. And so $a = -2$.

(b) The median is the point m such that

$$\begin{aligned} 1/2 &= \int_{\infty}^{m} f(x)\, \mathrm{d}x \\ &= \int_0^m (2 - 2x)\, \mathrm{d}x \\ &= \left[2x - x^2\right]_0^m \\ &= m^2 - 2m \end{aligned}$$

Hence $m^2 - 2m - 1/2 = 0$ and so

$$m = \frac{2 \pm \sqrt{4-2}}{2} = 1 \pm \frac{\sqrt{2}}{2}$$

But m cannot be bigger than 1, so $m = 1 - \sqrt{2}/2$.