Mathematics 101 — Solutions to First Midterm

Question 1 — 6 marks

Compute the following integrals.

(a) $\displaystyle\int_1^2 \left(x+\frac{1}{x}\right)^2 \,\mathrm{d}x$

(b) $\displaystyle\int_0^{\pi/4} \sec^2(t)\,\mathrm{d}t$

(c) $\displaystyle\int \frac{z}{\sqrt{1+z^2}}\,\mathrm{d}z$

Solution — 2 marks each

(a) Expand and integrate

$$\begin{aligned}\int_1^2 \left(x+\frac{1}{x}\right)^2 \mathrm{d}x &= \int_1^2 (x^2+2+x^{-2})\,\mathrm{d}x \\ &= [x^3/3+2x-1/x]_1^2 \\ &= (8/3+4-1/2)-(1/3+2-1) = 7/3+2+1/2 \\ &= (14+12+3)/6 = 29/6\end{aligned}$$

(b) Simple lookup

$$\begin{aligned}\int_0^{\pi/4} \sec^2 x\,\mathrm{d}x &= [\tan x]_0^{\pi/4} \\ &= (1-0) = 1\end{aligned}$$

(c) Substitution with $u = 1+z^2$, so $\mathrm{d}u = 2z\,\mathrm{d}z$

$$\begin{aligned}\int \frac{z}{\sqrt{1+z^2}}\,\mathrm{d}z &= \int \frac{z}{\sqrt{u}}\frac{\mathrm{d}u}{2z} \\ &= \int \frac{1}{2u^{1/2}}\,\mathrm{d}u \\ &= \sqrt{u}+c = \sqrt{1+z^2}+c\end{aligned}$$

Question 2 — 4 marks

Let $f(z) = \dfrac{\sec^2(z)}{(1+\tan(z))^q}$, where q is a real number.

(a) Compute $\displaystyle\int f(z)\, \mathrm{d}z$ for $q \neq 1$.

(b) Compute $\displaystyle\int f(z)\, \mathrm{d}z$ when $q = 1$.

Solution — 2 marks each

This is just a substitution integral with $u = 1 + tan(z)$, so $\mathrm{d}u = \sec^2(z)\, \mathrm{d}z$

- When $q \neq 1$, we have

$$\begin{aligned} \int \frac{\sec^2(z)}{(1+\tan(z))^q}\, \mathrm{d}z &= \int \frac{1}{u^q}\, \mathrm{d}u = \int u^{-q}\, \mathrm{d}u \\ &= \frac{1}{1-q} u^{1-q} + c \\ &= \frac{1}{1-q}(1+\tan(z))^{1-q} + c \end{aligned}$$

- This obviously breaks when $q = 1$, in that case we have

$$\begin{aligned} \int \frac{\sec^2(z)}{1+\tan(z)}\, \mathrm{d}z &= \int \frac{1}{u}\, \mathrm{d}u \\ &= \log|u| + c \\ &= \log|1+\tan(z)| + c \end{aligned}$$

Question 3 — 8 marks

(a) Let $f(x) = \dfrac{\sin(x)}{1+x^2}$. Compute $\displaystyle\int_{-\pi/2}^{\pi/2} f(x)\,\mathrm{d}x$.

(b) Let $f(x) = \displaystyle\int_1^{h(x)} e^{-t^2}\,\mathrm{d}t$. Compute $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ in terms of $h(x)$ and $h'(x)$.

Solution — 4 marks each

(a) This is obviously nasty — there is a trick. The function is odd.

$$\begin{aligned} f(-x) &= \frac{\sin(-x)}{1+(-x)^2} \\ &= \frac{-\sin(x)}{1+x^2} = -f(x) \end{aligned}$$

Since we are integrating an odd function from $-a$ to a, the answer is zero.

(b) This is just like we did in class. Let $G(t)$ be an anti-derivative of e^{-t^2}. Then we have

$$\begin{aligned} f(x) &= \int_1^{h(x)} e^{-t^2}\,\mathrm{d}t \\ &= G(h(x)) - G(1) \end{aligned}$$

So its derivative is (using the chain rule)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} f(x) &= \frac{\mathrm{d}}{\mathrm{d}x} G(h(x)) - \frac{\mathrm{d}}{\mathrm{d}x} G(1) \\ &= \frac{\mathrm{d}}{\mathrm{d}u} G(u) \cdot \frac{\mathrm{d}h}{\mathrm{d}x} - 0 \\ &= G'(u)h'(x) = \exp(-h(x)^2)h'(x) \end{aligned}$$

Question 4 — 10 marks

Let R be the region between the curve $y = e^x$ and the x-axis for $0 \leq x \leq 2$.

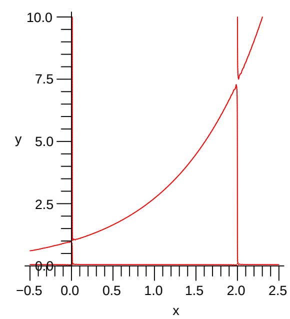
(a) Sketch R and compute its area.

(b) Write down the definite integral that gives the volume of the solid obtained by rotating R about the line $y = -3$. **Do not evaluate the integral**.

(c) Write down the definite integral that gives the volume of the solid obtained by rotating R about the y-axis. **Do not evaluate the integral**.

Solution — 4,4,2 marks

(a) The plot



Its area is simply

$$A = \int_0^2 e^x \, \mathrm{d}x$$
$$= [e^x]_0^2 = e^2 - 1$$

(b) Rotating line about $y = -3$ is using little disks. Each disk has radius $3 + e^x$ and width Δx. Its volume is $\pi(3 + e^x)^2 \Delta x$. However, there is a hole in each disk of radius 3. Hence the volume is

$$V = \int \pi(r^2 - 3^2) \, \mathrm{d}x$$
$$= \int_0^2 \pi((3 + e^x)^2 - 3^2) \, \mathrm{d}x$$

(c) Rotating around the y-axis is using cylindrical shells. Each shell has radius x, and height e^x. The volume of each shell is $2\pi rh = 2\pi x e^x$. Hence the total volume is

$$V = \int_0^2 2\pi rh \, \mathrm{d}r$$
$$= \int_0^2 2\pi x e^x \, \mathrm{d}x$$

Question 5 — 12 marks

Dodgy Brothers Waste Disposal Incorporated discovers that they have a huge conical tank completely full of toxic liquid waste sitting in their basement which they need to get rid of very quickly. In order to do this they decide to transfer the waste from the tank into the local sewer. Unfortunately the sewer is 10m above the top of the tank so they will have to pump it out.

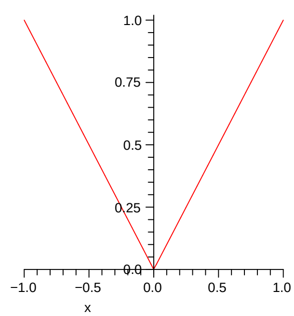
- The tank forms a circular cone which has a diameter of 2m at its top and is 1m deep.
- The toxic waste has the same mass as water — $1m^3 = 1000kg$.
- You may assume the gravitation constant is $g = 10N/kg$.

(a) Compute the volume of the tank using the method of cylindrical shells.

(b) Compute the work done pumping the toxic waste into the sewer.

Solution — 5,7 marks

(a) The tank is 2m in diameter and 1m deep. Hence it looks like



- The edge of the tank is described by the function $y = x$. A shell at x has radius x and height $1 - x$.
- Its volume is $2\pi x(1 - x)\Delta x$.
- Hence the volume of the tank is

$$\begin{aligned} V &= \int_0^1 2\pi(x - x^2)\,\mathrm{d}x \\ &= 2\pi[x^2/2 - x^3/3]_0^1 = 2\pi(1/2 - 1/3 - 0) = \pi/3 \cdot m^3 \end{aligned}$$

(b) We compute the work slice-by-slice.

- Let y measure the depth (in metres) from the bottom of the tank.
- A slice at depth y, has radius y, and so volume $\pi y^2 \Delta y$.
- The weight of the slice is $1000\pi y^2 \Delta y$ and so the force due to gravity on the slice is therefore $10000\pi y^2 \Delta y$.
- A slice at depth y needs to be pumped up $11 - y$ metres.

- Hence the total work is

$$\begin{aligned}W &= \int_0^1 10000\pi y^2(11-y)\,\mathrm{d}y = \int_0^1 10000\pi(11y^2-y^3)\,\mathrm{d}y \\ &= 10000\pi[11y^3/3 - y^4/4]_0^1 \\ &= 10000\pi(11/3-1/4) \\ &= \frac{41\cdot 10000}{12}\pi J\end{aligned}$$