Comparison test

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Comparing integrals

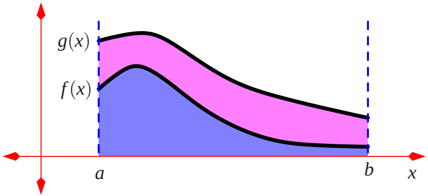
- Sometimes it is enough to just get an idea of the value of an integral.
- If the integrand is positive then we can use the following.

Comparing integrals

Consider two continuous functions f and g.

- If $0 \leq f(x) \leq g(x)$ for $a \leq x \leq b$, then

$$0 \leq \int_a^b f(x)\mathrm{d}x \leq \int_a^b g(x)\mathrm{d}x$$



The comparison test

The comparison test is a quick way of deciding if an improper integral is divergent or not.

Comparing integrals

Consider two continuous functions f and g.

- If $0 \leq f(x) \leq g(x)$ for $x \geq a$ we have

$$0 \leq \int_a^b f(x)\mathrm{d}x \leq \int_a^b g(x)\mathrm{d}x$$

It follows that:

- if $\int_a^\infty g(x)\mathrm{d}x$ is convergent then so is $\int_a^\infty f(x)\mathrm{d}x$.
- if $\int_a^\infty f(x)\mathrm{d}x$ is divergent then so is $\int_a^\infty g(x)\mathrm{d}x$.

A warmup example

Does the following integral converge: $\displaystyle\int_1^\infty e^{-x^2}\,\mathrm{d}x$

Now:

- $x^2 \geq x$ for $x \geq 1$, so
- $0 \leq e^{-x^2} \leq e^{-x}$ for $x \geq 1$, so

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$$0 \leq \int_1^{\infty} e^{-x^2}\,\mathrm{d}x \leq \int_1^{\infty} e^{-x}\,\mathrm{d}x$$

$$= \lim_{b\to\infty} \int_1^{b} e^{-x}\,\mathrm{d}x$$

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$$0 \leq \int_1^\infty e^{-x^2} \mathrm{d}x \leq \int_1^\infty e^{-x} \mathrm{d}x$$
$$= \lim_{b\to\infty} \int_1^b e^{-x} \mathrm{d}x$$
$$= \lim_{b\to\infty} \left[-e^{-x}\right]_1^b$$
$$= e^{-1} - \lim_{b\to\infty} e^{-b} = e^{-1}$$

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0 \leq \int_1^\infty e^{-x^2}\mathrm{d}x \leq \int_1^\infty e^{-x}\mathrm{d}x \\
&= \lim_{b\to\infty}\int_1^b e^{-x}\mathrm{d}x \\
&= \lim_{b\to\infty}\left[-e^{-x}\right]_1^b \\
&= e^{-1} - \lim_{b\to\infty} e^{-b} = e^{-1}
\end{aligned}
$$

The integral is finite — it is actually about $0.1394027925\ldots$

An example

Does the following integral converge: $\int_1^{\infty} \cos^2 x \cdot e^{-x^2} \mathrm{d}x$

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- $0 \leq \cos^2 x \leq 1$ everywhere.
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The integral is finite — it is actually about 0.0181648382 . . .

Let us extend that example

Does the following integral converge: $\displaystyle\int_0^\infty \cos^2 x \cdot e^{-x^2} \mathrm{d}x$

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- We need to fix up $0 \leq x \leq 1$.

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- Split the interval — we need to bound $\int_0^1 \cos^2 x \cdot e^{-x^2} \mathrm{d}x$.

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- We know that $0 \leq e^{-x^2} \leq 1$ and $0 \leq \cos^2 x \leq 1$, so

$$0 \leq \int_0^1 \cos^2 x \cdot e^{-x^2} \mathrm{d}x \leq \int_0^1 1 \mathrm{d}x$$

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- This is finite — the integral converges.

Another example

Does the following integral converge: $\int_e^{\infty} (\log x)^{3/2} \, \mathrm{d}x$

Now

- for $u > 1$ we have $u^{3/2} > u$ (but not when $0 \leq u \leq 1$).
- so $(\log x)^{3/2} > \log x > 0$ (for $x > e$).

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$$\int_e^{\infty} (\log x)^{3/2}\,\mathrm{d}x \geq \int_e^{\infty} \log x\,\mathrm{d}x = \lim_{b\to\infty} \int_e^{b} \log x\,\mathrm{d}x$$
$$= \lim_{b\to\infty} \left[x \log x - x\right]_e^b$$

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\int_e^\infty (\log x)^{3/2}\,\mathrm{d}x \ge \int_e^\infty \log x\,\mathrm{d}x &= \lim_{b\to\infty} \int_e^b \log x\,\mathrm{d}x \\
&= \lim_{b\to\infty} \left[x\log x - x\right]_e^b \\
&= \lim_{b\to\infty} \left(b\left(\log b - 1\right)\right) - 0 \\
&= \text{divergent}
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So the integral is divergent.

Yet another example

Does the following converge: $\displaystyle\int_1^\infty \frac{4}{2x + 17x^5}\mathrm{d}x$

Now

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- So $4/(2x + 17x^5) < 4/17x^5$.

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Hence

$$0 \leq \int_1^\infty \frac{4}{2x + 17x^5} \mathrm{d}x \leq \frac{4}{17} \int_1^\infty \frac{1}{x^5} \mathrm{d}x = \frac{1}{17}$$

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Hence

$$0 \leq \int_1^\infty \frac{4}{2x+17x^5}\mathrm{d}x \leq \frac{4}{17}\int_1^\infty \frac{1}{x^5}\mathrm{d}x = \frac{1}{17}$$

So the integral converges — it is $\frac{1}{2}\log\left(\frac{19}{17}\right) = 0.055612818\ldots$