1 Homework 7 - Answers

The total marks = 18

1. **4 marks True:**

Result. Let $A, B, C$ be sets. Then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. We prove each set is a subset of the other.

Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. Since $x \in B \cup C$, we have $x \in B$ or $x \in C$. If $x \in B$, then $x \in A \cap B$. If $x \in C$, then $x \in A \cap C$. In either case, we have $x \in (A \cap B) \cup (A \cap C)$. Hence $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Now let $x \in (A \cap B) \cup (A \cap C)$. Then $x \in A \cap B$ or $x \in A \cap C$. If $x \in A \cap B$ then $x \in A$ and $x \in B$. Since $x \in B$, $x \in B \cup C$. Thus, $x \in A \cap (B \cup C)$. Similarly, if $x \in A \cap C$ then $x \in A$ and $x \in C$. Since $x \in C$, $x \in B \cup C$. Thus, $x \in A \cap (B \cup C)$. This shows $A \cap (B \cup C) \supseteq (A \cap B) \cup (A \cap C)$.

2. **2 marks True**

Result. Let $A$ and $B$ be sets, then $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

Proof. Let $A$ and $B$ be sets. Let $X \in \mathcal{P}(A) \cup \mathcal{P}(B)$. Then $X \in \mathcal{P}(A)$ or $X \in \mathcal{P}(B)$. If $X \in \mathcal{P}(A)$, then $X \subseteq A$, so $X \subseteq A \cup B$. If $X \in \mathcal{P}(B)$, then $X \subseteq B$, so $X \subseteq A \cup B$. Hence, in either case $X \in \mathcal{P}(A \cup B)$.

3. **2 marks False:** The statement “Let $A$ and $B$ be sets, $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$” is false.

Disproof. Let $A = \{1\}$, $B = \{2\}$. Then $\{1, 2\} \in \mathcal{P}(A \cup B)$ but since $\{1, 2\} \not\subseteq A$ and $\{1, 2\} \not\subseteq B$, we have $\{1, 2\} \not\in \mathcal{P}(A) \cup \mathcal{P}(B)$.

4. Not marked (sorry).

False: The statement “If $A, B, C$ are sets, then $A - (B \cup C) = (A - B) \cup (A - C)$” is false.

Disproof. Let $A = \{1, 2\}$, $B = \{1\}$, $C = \emptyset$. Then $A - (B \cup C) = \{1, 2\} - \{1\} = \{2\}$, but $(A - B) \cup (A - C) = (\{1, 2\} - \{1\}) \cup (\{1, 2\} - \emptyset) = \{2\} \cup \{1, 2\} = \{1, 2\}$.

5. **2 marks False:** The statement “Suppose $A, B, C$ are sets. If $A = B - C$, then $B = A \cup C$” is false.

Disproof. Let $B = \{1\}$, $C = \{2\}$. Then $A = B - C = \{1\}$ and $A \cup C = \{1, 2\} \neq B$.

6. Not marked (sorry).

False: The statement “Let $X, A, B$ be sets. If $X \subseteq A \cup B$, then $X \subseteq A$ or $X \subseteq B$” is false.
Disproof. Let $A = \{1\}$, $B = \{2\}$, $X = \{1, 2\}$. Then $X \subseteq A \cup B$ but $X \not\subseteq A$ and $X \not\subseteq B$.

7. **4 marks** True:

**Result.** If $A, B, C$ are sets $(A \cup B) - C = (A - C) \cup (B - C)$.

**Proof.** Let $A, B, C$ be sets. We prove each set is a subset of the other.
Let $x \in (A \cup B) - C$, then $x \in A \cup B$ and $x \notin C$. Since $x \in A \cup B$, we have $x \in A$ or $x \in B$. If $x \in A$, then $x \in A - C$. Similarly, if $x \in B$, then $x \in B - C$. Hence in either case, $x \in (A - C) \cup (B - C)$.

Now let $x \in (A - C) \cup (B - C)$. Then $x \in A - C$ or $x \in B - C$. If $x \in A - C$, then $x \in A$ and $x \notin C$. Since $x \in A$, we have $x \in A \cup B$. Similarly, if $x \in B - C$, then $x \in B$ and $x \notin C$. Since $x \in B$ we have $x \in A \cup B$. Hence in either case, $x \in A \cup B$ and $x \notin C$. Thus, $x \in (A \cup B) - C$.

8. **2 marks** True:

**Result.** Let $k, \ell \in \mathbb{Z}$ and let $A = \{x \in \mathbb{Z} : k\ell \mid x\}$, $B = \{x \in \mathbb{Z} : k \mid x\}$, and $C = \{x \in \mathbb{Z} : \ell \mid x\}$. Then $A \subseteq B \cap C$.

**Proof.** Let $x \in A$. Then $k\ell \mid x$, so we can write $x = (k\ell)n$ for some $n \in \mathbb{Z}$. Then $x = k(\ell n)$, and since $\ell n \in \mathbb{Z}$, we have $k \mid x$. Hence $x \in B$. Also $x = \ell (kn)$ and since $kn \in \mathbb{Z}$, we have $\ell \mid x$. Hence $x \in C$. Thus, $x \in B \cap C$. So we have $A \subseteq B \cap C$.

9. **2 marks** False: Let $k, \ell \in \mathbb{Z}$ and let $A = \{x \in \mathbb{Z} : k\ell \mid x\}$, $B = \{x \in \mathbb{Z} : k \mid x\}$, and $C = \{x \in \mathbb{Z} : \ell \mid x\}$. The statement “$A = B \cap C$” is false.

Disproof. Let $k = 4$ and $\ell = 6$, so $A = \{x \in \mathbb{Z} : 24 \mid x\}$, $B = \{x \in \mathbb{Z} : 4 \mid x\}$ and $C = \{x \in \mathbb{Z} : 6 \mid x\}$. Since 12 is divisible by both 4 and 6 we have $12 \in B \cap C$, but $12 \notin A$.

Note that the disproof works for many values of $k, \ell$. For example if you take $k = \ell = 2$, then $2 \in B \cap C$ but $2 \notin A$.

**Some Comments/Tips/Explanations**

**Question 1**

We prove this result from first principles (from definitions, axioms, and immediate consequences of the definitions). To prove two sets are equal we always follow the same basic set up. We need to prove each set is a subset of the other, so our proof has two parts. In each part we are proving a statement of form $X \subseteq Y$. To do this we take an arbitrary element from $X$ and prove it is an element of $Y$. 

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Question 2

In this question we are again proving that one set is a subset of another. So we start by taking an element of the first set then prove that it belongs to the other. We need to know the definition of power set to complete this question.

\[ P(A) = \{ X : X \subseteq A \} \]

Question 3

Note, this is really the statement \( \forall A, B, C, P(A \cup B) \subseteq P(A) \cup P(B) \). The negation is \( \exists A, B, C, P(A \cup B) \not\subseteq P(A) \cup P(B) \). So all we need to do to disprove the statement is give an example of sets \( A, B, C \) with \( P(A \cup B) \not\subseteq P(A) \cup P(B) \).

In all of these questions, a good strategy is to try a couple simple examples (or look at Venn diagrams if possible) to see if you think the statement is true. If you find a counterexample, the statement is false and you are done. If after trying a couple of examples, you think the statement is true, try to prove it. You will either be able to write up the proof, or you will run into a problem that helps you come up with a counterexample.

Question 4

See the strategy from Question 3. Venn Diagrams are quite useful for this question. If we draw the diagram carefully, we can see that any element of \( A \cap B \) that is not in \( C \) is an element of the right hand side, but not the left hand side. We construct an example using this observation.

Question 5

Again, a Venn Diagram helps here. We can see that whenever \( C \) is not a subset of \( B \) we get a counterexample.

Question 6

Similar to questions 3-5.

Question 7

Since the instructions ask you to prove from first principles, you cannot use distributive laws, etc. for this question. For the general strategy, see the explanation of Question 1.

Question 8

This question uses the same strategy as Question 2. We show any element of \( A \) belongs to both \( B \) and \( C \), using the definitions of each of the sets and the definition of divides.
Question 9

This statement has the form “∀k, ℓ ∈ ℤ . . .”, so the negation is ∃k, ℓ ∈ ℤ . . . . This means to disprove the statement we only need to give an example of k and ℓ for which the statement fails. Any choice of k and ℓ that have a common divisor will work here.

Be careful when you are considering statements like this. Many people will try to prove the result using the argument “if ℓ | x and k | x then kℓ | x”, but this statement is false! This is why it is important to make sure we justify everything carefully.