Homework 2 — Answers

The total marks = 24

1. [3 marks]

Result. Suppose \( x, y \in \mathbb{Z} \). If \( x \) and \( y \) are odd then \( xy \) is odd.

Proof. Let \( x, y \in \mathbb{Z} \) be odd. Then by definition \( x = 2k + 1 \) and \( y = 2\ell + 1 \) for some \( k, \ell \in \mathbb{Z} \). So we have

\[
xy = (2k + 1)(2\ell + 1) = 4k\ell + 2k + 2\ell + 1 = 2(k\ell + k + \ell) + 1.
\]

Since \( k\ell + k + \ell \) is an integer, this implies \( xy \) is odd.

2. [3 marks]

Result. Suppose \( a, b, c \in \mathbb{Z} \). If \( a \mid b \) and \( a \mid c \), then \( a \mid (b + c) \).

Proof. Let \( a, b, c \in \mathbb{Z} \). Suppose \( a \mid b \) and \( a \mid c \). So by the definition of divisibility, \( b = ak \) and \( c = a\ell \) for some \( k, \ell \in \mathbb{Z} \). Then

\[
b + c = ak + a\ell = a(k + \ell).
\]

Hence, since \( k \) and \( \ell \) are integers, we have \( a \mid (b + c) \).

3. [3 marks]

Result (Question 3). Suppose \( a \) is an integer. If \( 5 \mid 2a \), then \( 5 \mid a \).

Proof. Suppose \( a \) is an integer and \( 5 \mid 2a \). Then by definition there is some integer \( n \) such that

\[
5n = 2a.
\] (1)

It follows that \( 5n \) is even. But then \( n \) cannot be odd since we know from Question 1 the product of two odd numbers is odd. Hence \( n \) must be even, so \( n = 2m \) for some integer \( m \). Plugging this into equation 1, we have

\[
5(2m) = 2a.
\]

Then dividing by 2, we get

\[
5m = a.
\]

Since \( m \) was an integer, this shows \( 5 \mid a \).

There is another way to prove this result, but it relies on the following lemma. If you use this argument you would also need to prove the lemma in order to get full credit. This lemma is actually exercise 7.29 in the book. You can find the proof on page 269.
Lemma. Let $p$ be a prime number. If $p|ab$ then $p|a$ or $p|b$.

Proof of result. Suppose $a$ is an integer and $5 \mid 2a$. Then since 5 is prime and $5 \nmid 2$, we must have $5 \mid a$.

4. [3 marks]

Result. Suppose $x$ and $y$ are positive real numbers. If $x < y$, then $x^2 < y^2$.

Proof. Suppose $x$ and $y$ are positive real numbers and $x < y$. Then

\begin{align*}
x < y & \quad \text{subtract $x$ from both sides} \\
0 < y - x & \quad \text{multiply both sides by $x + y$ which is positive} \\
0 < (y - x)(y + x) & \quad \text{expand it} \\
0 < y^2 - x^2 & \quad \text{add $x^2$ to both sides} \\
x^2 < y^2 &
\end{align*}

as required.

5. [3 marks]

Result (Question 5(a)). Let $n$ be a nonzero integer. If $n^2 \mid n$, then $n = -1$ or $n = 1$.

Proof. Suppose $n^2 \mid n$. Then there is some integer $k$ such that

$$n^2k = n.$$  

Dividing both sides by $n$ we see

$$nk = 1.$$  

This tells us that $n$ is a divisor of 1. Since the only divisors of 1 are 1 and $-1$, we must have $n = 1$ or $n = -1$.

6. [3 marks]

Result. If $n$ is an odd integer, then $4 \mid ((n - 1)(n + 1))$.

Proof. Let $n$ be an odd integer, so $n = 2k + 1$ for some $k \in \mathbb{Z}$. Then

\begin{align*}
(n - 1)(n + 1) &= (2k + 1 - 1)(2k + 1 + 1) \\
&= (2k)(2k + 2) \\
&= 4k^2 + 4k = 4(k^2 + k)
\end{align*}

Since $k^2 + k$ is an integer, this implies $4 \mid (k^2 + k)$.

7. [3 marks]

2
Result. Let $x \in \mathbb{R}$. If $x < 0$, then $x + \frac{1}{x} < -1$.

Proof. Suppose $x < 0$. Since $x + 1$ is a real number, we know $0 \leq (x+1)^2$. Then, since $x < 0$, and $0 \leq (x+1)^2$ we have

$$
x < (x + 1)^2 \quad \text{expand}
x < x^2 + 2x + 1 \quad \text{subtract 2x from both sides}
-x < x^2 + 1 \quad \text{divide by } x \text{ which is negative}
-1 > x + \frac{1}{x}
$$

as required. Note, that when we divide the inequality by $x$ it changes sign since $x$ is a negative number.

8. 3 marks

Result. If $n \in \mathbb{N}$ is not prime, then $2^n - 1$ is not prime.

Proof. Suppose $n$ is not prime. Then either $n = 1$ or $n > 1$. If $n = 1$ then $2^n - 1 = 2 - 1 = 1$ which is not prime.

So now assume that $n > 2$ is not prime and so is divisible by a natural number other than $n$ and 1. So we can write $n = k\ell$ where $k, \ell \in \mathbb{N}$ and $1 < k < n$. This, in turn, implies that $1 < \ell < n$.

Now (using the hint in the statement of the question), we can write

$$2^n - 1 = 2^{k\ell} - 1 = (2^\ell)^k - 1 = (2^\ell)^k - 1^k \quad \text{using the hint with } x = 2^\ell, y = 1 = (2^\ell - 1)(2^{\ell(k-1)} + 2^{\ell(k-2)} + \cdots + 2^\ell + 1)$$

Now since $\ell > 1$, both factors in the above expression are strictly larger than 1. Hence the number $2^n - 1$ is divisible by a number other than 1 and $2^n - 1$. Thus $2^n - 1$ is not prime.
Some explanations/scratch work/tips

General Strategy
All of these results are implications, so this next bit of reasoning applies to every problem. Thinking about our truth table for implications we know that

- If the hypothesis is false, the implication is true. So there is no work to do in this case.
- If the hypothesis is true, then the implication can be true or false, depending on the truth value of the conclusion. So this is where we have to do some work. We need to show the conclusion is true.

We will use direct proof to prove each of these statements. So we assume the hypothesis is true and our goal is to show the conclusion must then also be true.

Now onto some of the questions...
- (Question 1) We know we need to assume the hypothesis

  \[ x \text{ and } y \text{ are odd integers.} \]

  and prove the conclusion

  \[ xy \text{ is an odd integer.} \]

The first step is to use the definition(s) involved to flesh out the statements.

  - Hypothesis: \( x = 2k + 1 \) and \( y = 2\ell + 1 \) for some \( k, \ell \in \mathbb{Z} \)
  - Conclusion: \( x = 2m + 1 \) for some \( m \in \mathbb{Z} \)

Now we notice, using our hypothesis

\[ xy = (2k + 1)(2\ell + 1) = 4k\ell + 2k + 2\ell + 1 \]

and with a bit more work we can get this into the form we’re looking for

\[ xy = (2k + 1)(2\ell + 1) = 4k\ell + 2k + 2\ell + 1 = 2(2k\ell + k + \ell) + 1. \]

Now that we have all our ideas together, we can do the write-up (above). Make sure you start from the hypothesis, end with the conclusion, and each step follows logically from the step(s) before it.

- (Question 2) This is similar to question 1. The key is to use the definition of divides to flesh out the hypothesis and conclusion.
• (Question 3) Similar to questions 1 and 2, you should start by using the definitions. After using the definitions we know we can write \(5n = 2a\) and we would like to show we can write \(5k = a\). It is not immediately clear what happens when we divide by 2, so this proof is not as simple as the first two.

To proceed from here we need to think about what other ideas and results might help. We notice that the number \(5n = 2a\) is even by definition, so we can apply what we know about even/odd numbers to push this further and eventually to get the proof given above.

• (Question 4) In order to prove an inequality, we often work in reverse. Starting with what we’re trying to prove we can manipulate the equation as follows.

\[
\begin{align*}
y^2 &> x^2 \\
y^2 - x^2 &> 0 \quad \text{subtract } x^2 \\
(y - x)(y + x) &> 0 \quad \text{factor} \\
y - x &> 0 \quad \text{divide by the positive number } y+x \\
y &> x \quad \text{add } x
\end{align*}
\]

Since we know \(x < 0\) and \(0 \leq (x + 1)^2\), we know this last inequality is true! So we are done with our scratch work. To finish we rewrite our work in the opposite order: starting with what we know and ending with what we are trying to prove. When you do this it is important to make sure you can justify each step!

If you get stuck try writing both inequalities in multiple ways and seeing how they are related (in this case \(y - x\) is a factor of \(y^2 - x^2\)).

• (Question 5) As in questions 1 and 2, the key here is to use the definition(s) carefully. Only the hypothesis needs to be rephrased here, but once you do that you can see that the equation you get, \(n^2k = n\), easily simplifies to \(nk = 1\). The conclusion follows then immediately.

• (Question 6) As in question 4 it is helpful to work in reverse in our scratch work.

\[
\begin{align*}
x + \frac{1}{x} &< -1 \\
x^2 + 1 &> -x \quad \text{multiply by } x \\
x^2 + 2x + 1 &> x \quad \text{add } 2x \text{ to both sides} \\
(x + 1)^2 &> x
\end{align*}
\]

How did we come up with this?

- Pay close attention to any hints. The first step was given as a hint on the homework document.
Think about what you know and how you will use it. In the second step if we had moved every term to one side we get the expression $x^2 + x + 1$. This reminds me of two things: this is close to the expansion of $(x + 1)^2$ and the square of any real number is nonnegative. I realize maybe these pieces of information will help so I see where they lead.

Working with questions like this takes patience. Don’t get discouraged if your first couple of attempts don’t work.

* (Question 7) As above, the first thing you need to do in this question is look at the definitions involved.

**Definition.** A natural number is **prime** if it has exactly 2 positive divisors, 1 and $n$.

This means a natural is not prime if it has more than 2 divisors (or only one divisor, as in the case of $n = 1$). Using this definition we rephrase the hypothesis and conclusion.

- Since $n$ is not prime, we know $n = 1$ or $n = \ell m$ for $\ell$ and $m$ not equal to 1 and $n$.
- To prove $2^n - 1$ is not prime we need to show $2^n - 1 = cd$ for some integers $c, d$ not equal to 1 or $2^n - 1$ (or that $2^n - 1$ equals 1, which happens only in the case $n = 1$)

Again, pay attention to the hint! We know we are looking for a way to factor $2^n - 1$ and this hint shows us how to factor $a^k - b^k$. (Note, I avoided using $a, b, k$ above because I knew I wanted to use this formula later, and I didn’t want to get confused.)

We can try using this formula with $a = 2$, $b = 1$, $k = n$. But in this case the first factor is $2 - 1 = 1$, so we do not get a nontrivial factorization. :(

However, if we substitute $n = \ell m$ we can use this formula with $a = 2^\ell$, $b = 1$, and $k = m$. Since $\ell$ is not equal to 1 or $n$, $2^\ell - 1$ is not equal to 1 or $2^n - 1$, hence $2^n - 1$ has more than 2 divisors.