Mathematics 220 — Midterm — 45 minutes

October 17th 2018

- The quiz consists of 6 pages and 5 questions worth a total of 23 marks.
- This is a closed-book examination. None of the following are allowed: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- No work on this page will be marked.
- Fill in the information below before turning to the quiz.

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1. 5 marks
(a) Write the negation of the following statement:
“For every \( n \in \mathbb{Z} \), \((n + 1)^2 \geq 9\) or \( n < 2 \).”

Solution: \( \exists n \in \mathbb{Z} \) s.t. \((n + 1)^2 < 9\) and \( n \geq 2 \).

(b) Write the negation of the following statement:
“There exists \( x \in \mathbb{Z} \) so that for every \( r \in \mathbb{Q} \), if \( x \neq 0 \) then \( rx = r \).”

Solution: \( \forall x \in \mathbb{Z}, \exists r \in \mathbb{Q} \) s.t. \( x \neq 0 \) and \( rx \neq r \).

(c) Write the converse and contrapositive of the following statement:
“If you do not like chocolate, then you do not like brownies.”

Be sure to indicate which is which.

Solution:
- converse = “If you do not like brownies, then you do not like chocolate.”
- contrapositive = “If you like brownies, then you like chocolate.”

(d) Let \( a \in \mathbb{N} \) with \( a \geq 2 \). Carefully define what “\( a \) is prime” means.

Solution: We say \( a \) is prime if \( a \) has exactly two distinct divisors in \( \mathbb{N} \), namely 1 and \( a \).
2. [5 marks] Let $n$ be an integer. Prove or disprove the following two statements
(a) If $3 
mid n$, then $3 \mid (n^2 - 6n + 5)$.

**Solution:** This is true.

*Proof.* Suppose $3 \nmid n$, then $n = 3k + 1$ for some $k \in \mathbb{Z}$ or $n = 3k + 2$ for some $k \in \mathbb{Z}$.

- Assume $n = 3k + 1$ for some $k \in \mathbb{Z}$. Hence

$$n^2 - 6n + 5 = 9k^2 - 12k = 3(3k^2 - 4k)$$

Since $3k^2 - 4k \in \mathbb{Z}$, it follows that $3 \mid n^2 - 6n + 5$.

- Assume $n = 3k + 2$ for some $k \in \mathbb{Z}$. Hence

$$n^2 - 6n + 5 = 9k^2 - 6k - 3 = 3(3k^2 - 2k - 1)$$

Since $3k^2 - 2k - 1 \in \mathbb{Z}$, it follows that $3 \mid n^2 - 6n + 5$.

(b) If $3 \nmid n^2$ then $3 \mid (n^2 + 2n)$.

**Solution:** This is false. Let $n = 2$, then $n^2 + 2n = 8$ which is not divisible by 3.
3. **3 marks** Prove the following statement

Let $x \in \mathbb{R}$. If $x > 0$ then \( \frac{2x}{x+1} \leq \frac{x+1}{2} \).

**Solution:** Assume that $x$ is a positive real.

\[
(x - 1)^2 \geq 0 \quad \text{since the square of any real number is non-negative}
\]

\[
x^2 - 2x + 1 \geq 0 \quad \text{expanding the left hand side}
\]

\[
x^2 + 2x + 1 \geq 4x \quad \text{adding } 4x \text{ to both sides}
\]

\[
(x + 1)^2 \geq 4x \quad \text{factoring the left hand side}
\]

\[
\frac{x + 1}{2} \geq \frac{2x}{x+1} \quad \text{multiplying both sides by } \frac{1}{2(x+1)} > 0
\]
4. **4 marks** Prove the following statement.

For any integer \( n \geq 1, \)

\[
\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \leq 2 - \frac{2}{(n+1)!}.
\]

Recall that \( n! = 1 \times 2 \times 3 \times \cdots \times n \)

**Solution:** We prove the statement using induction.

- **Base case.** Since \( 1 = 2 - \frac{2}{2!}, \) the statement is true when \( n = 1. \)

- **Inductive step.** We assume that

\[
\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!} \leq 2 - \frac{2}{(k+1)!}.
\]

Adding \( \frac{1}{(k+1)!} \) to both sides gives

\[
\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!} + \frac{1}{(k+1)!} \leq 2 - \frac{2}{(k+1)!} + \frac{1}{(k+1)!}.
\]

Consider the right-hand side

\[
2 - \frac{2}{(k+1)!} + \frac{1}{(k+1)!} = 2 - \frac{1}{(k+1)!} = 2 - \frac{k+2}{(k+2)!} = 2 - \frac{k}{(k+2)!} - \frac{2}{(k+2)!} \leq 2 - \frac{2}{(k+2)!} \text{ since } \frac{k}{(k+2)!} \geq 0.
\]

Hence

\[
\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!} + \frac{1}{(k+1)!} \leq 2 - \frac{2}{(k+2)!}
\]

as required.

Hence, by mathematical induction the statement is true for all integers \( n \geq 1. \)
5. **6 marks** Let $S = \{n \in \mathbb{Z} : 3 \nmid n\}$.
Determine whether the following three statements are true or false — explain your answers (“true” or “false” is not sufficient).

(i) $\exists x \in S$ s.t. $\exists y \in S$ s.t. $x + y \in S$.
(ii) $\exists x \in S$ s.t. $\forall y \in S, x + y \in S$.
(iii) $\forall x \in S, \exists y \in S$ s.t. $x + y \in S$.

**Solution:**

(i) This is true. Let $x = 1, y = 1$ then $x + y = 2 \in S$ as required.

(ii) False — negation is

\[
\forall x \in S, \exists y \in S \text{ s.t. } x + y \notin S
\]

Let $x \in S$, then either $x = 3k + 1$ for some $k \in \mathbb{Z}$ or $x = 3k + 2$ for some $k \in \mathbb{Z}$. We consider these cases separately.

- If $x = 3k + 1$ for some $k \in \mathbb{Z}$, take $y = 2$ then $x + y = 3k + 3 \notin S$.
- If $x = 3k + 2$ for some $k \in \mathbb{Z}$, take $y = 1$ then $x + y = 3k + 3 \notin S$.

(iii) This is true. Let $x \in S$, then either $x = 3k + 1$ for some $k \in \mathbb{Z}$ or $x = 3k + 2$ for some $k \in \mathbb{Z}$. We consider these cases separately.

- If $x = 3k + 1$ for some $k \in \mathbb{Z}$, take $y = 1$ then $x + y = 3k + 2 \in S$.
- If $x = 3k + 2$ for some $k \in \mathbb{Z}$, take $y = 2$ then $x + y = 3k + 4 = 3(k + 1) + 1 \in S$. 
