Mathematics 220 — Midterm — 45 mintues

October 17th 2018

• The quiz consists of 6 pages and 5 questions worth a total of 23 marks.

• This is a closed-book examination. None of the following are allowed: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

• No work on this page will be marked.

• Fill in the information below before turning to the quiz.

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1. **5 marks**

(a) Write the negation of the following statement:

“For every \( n \in \mathbb{Z} \), \( n \leq 1 \) or \( n^2 \geq 4 \).”

**Solution:** \( \exists n \in \mathbb{Z} \) s.t. \( n > 1 \) and \( n^2 < 4 \).

(b) Write the negation of the following statement:

“For every \( x \in \mathbb{Q} \), there exists \( r \in \mathbb{R} \) so that if \( x \geq 0 \) then \( r^2 = x \).”

**Solution:** \( \exists x \in \mathbb{Q} \) so that \( \forall r \in \mathbb{R} \), \( x \geq 0 \) and \( r^2 \neq x \).

(c) Write the converse and contrapositive of the following statement:

“If you do not speak Korean then understanding K-pop is hard.”

Be sure to indicate which is which.

**Solution:**

- converse = “If understanding K-pop is hard, then you do not speak Korean.”
- contrapositive = “If understanding K-pop is easy, then you speak Korean.”

(d) Let \( a, b \in \mathbb{Z} \). Carefully define what “\( a \mid b \)” means.

**Solution:** We write \( a \mid b \) when we can write \( b = ka \) for some integer \( k \).
2. [5 marks] Let $n$ be an integer. Prove or disprove the following two statements

(a) If $n \in \mathbb{Z}$ then $n^2 + 4n + 8$ is even.

Solution: This is false. Let $n = 1$. Then $1 + 4 + 8 = 13$ which is odd.

(b) If $n \in \mathbb{Z}$ then $n^2 + 3n + 8$ is even.

Solution: This is true.

Proof. Let $n \in \mathbb{Z}$, then $n$ is even or odd.

- Assume $n$ is even and so $n = 2k$ for some $k \in \mathbb{Z}$. Hence

  \[n^2 + 3n + 8 = 4k^2 + 6k + 8 = 2(2k^2 + 3k + 4)\]

  Since $2k^2 + 3k + 8 \in \mathbb{Z}$, it follows that $n^2 + 3n + 8$ is even.

- Assume $n$ is odd and so $n = 2k + 1$ for some $k \in \mathbb{Z}$. Hence

  \[n^2 + 3n + 8 = 4k^2 + 4k + 1 + 6k + 3 + 8 = 2(2k^2 + 5k + 6)\]

  Since $2k^2 + 5k + 6 \in \mathbb{Z}$, it follows that $n^2 + 3n + 8$ is even.
3. **3 marks** Prove the following statement

Let \( x \in \mathbb{R} \). If \( x < 0 \) then \( \frac{x - 2}{5} \leq \frac{2x + 1}{2 - x} \).

**Solution:**

*Proof.* Assume that \( x \) is a negative real number. Then

\[
\begin{align*}
(x + 3)^2 & \geq 0 \quad \text{expand both sides} \\
x^2 + 6x + 9 & \geq 0 \quad \text{subtract } 10x + 5 \text{ from both sides} \\
x^2 - 4x + 4 & \geq -10x - 5 \quad \text{factor both sides} \\
(x - 2)^2 & \geq -5(2x + 1) \quad \text{divide by } 5(x - 2) < 0 \\
\frac{x - 2}{5} & \leq \frac{-2x + 1}{x - 2} \\
& = \frac{2x + 1}{2 - x}
\end{align*}
\]

as required.

Potential scratch work (for question difficulty sanity check):

\[
\begin{align*}
\frac{x - 2}{5} & \leq \frac{2x + 1}{2 - x} \quad (2 - x) > 0 \\
-(x - 2)^2 & \leq 5(2x + 1) \\
(x - 2)^2 & \geq -10x - 5 \\
x^2 - 4x + 4 & \geq -10x - 5 \\
x^2 + 6x + 9 & \geq 0 \\
(x + 3)^2 & \geq 0 \checkmark
\end{align*}
\]
4. **4 marks** Prove the following statement.

For any integer \( n \geq 1 \),

\[
1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.
\]

**Solution:** We prove the statement using induction.

- **Base case.** Since \( 1 = 2 - \frac{1}{1} \), the statement is true when \( n = 1 \).

- **Inductive step.** We assume that

\[
1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k}.
\]

Adding \( \frac{1}{(k+1)^2} \) to both sides gives

\[
1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}.
\]

Consider the right-hand side

\[
2 - \frac{1}{k} + \frac{1}{(k+1)^2} \leq 2 - \frac{(1+k)^2 - k}{k(k+1)^2} = 2 - \frac{k(k+1) + 1}{k(k+1)^2}
\]

\[
\leq 2 - \frac{1}{k+1} - \frac{1}{(k+1)^2}
\]

\[
\leq 2 - \frac{1}{k+1} \quad \text{since} \quad \frac{1}{(k+1)^2} \geq 0.
\]

Hence

\[
1 + \frac{1}{4} + \cdots + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}
\]

as required.

Hence, by mathematical induction the statement is true for all integers \( n \geq 1 \).
5. [6 marks] Let $P = \{2, 3, 5, 7, \ldots \} \subset \mathbb{N}$ be the set of prime numbers. Determine whether the following three statements are true or false — explain your answers (“true” or “false” is not sufficient).

Hint — the number 2 is the only even prime number.

(i) $\exists x \in P$ s.t. $\exists y \in P$ s.t. $x + y \in P$.

(ii) $\exists x \in P$ s.t. $\forall y \in P, x + y \in P$.

(iii) $\forall x \in P, \exists y \in P$ s.t. $x + y \in P$.

Solution:

(i) This is true. Let $x = 2, y = 3$ then $x + y = 5 \in P$ as required.

(ii) False — negation is

$$\forall x \in P, \exists y \in \mathbb{P} \text{ s.t. } x + y \notin P$$

We split this into 2 cases; either $x = 2$ or $x \neq 2$. If $x = 2$ then set $y = 7$. The number $x + y = 9 = 3 \times 3 \notin P$. Otherwise $x$ must be an odd number, so set $y = 3$. Then $x + y$ is even, and so divisible by 2, and thus not in $P$.

(iii) False — negation is

$$\exists x \in P, \forall y \in \mathbb{P} \text{ s.t. } x + y \notin P$$

Take $x = 7$. Then if $y = 2, x + y = 9 \notin P$. And if $y \neq 2$, then $y$ is odd, so $y + 7$ is even, and hence not a prime number.