Homework 9

1. Define a relation $R$ on $\mathbb{Z}$ as $xRy$ if and only if $4 \mid (x + 3y)$.
   
   (a) Prove $R$ is an equivalence relation.
   
   (b) Describe its equivalence classes.

2. Consider the set $f = \{(x^3, x) : x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}$. Is this a function from $\mathbb{R}$ to $\mathbb{R}$? Explain your answer.

3. Consider the set $h = \{((x, y), (3y, 2x, x + y)) : x, y \in \mathbb{R}\}$. Is $h$ a function? If so, what is its domain, codomain and range?

4. A function $f : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is defined as $f(n) = (2n, n + 3)$.
   
   (a) Prove or disprove: The function $f$ is injective.
   
   (b) Prove or disprove: The function $f$ is surjective.

5. Let $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined as $g(m, n) = 3n - 4m$.
   
   (a) Prove or disprove: The function $g$ is injective.
   
   (b) Prove or disprove: The function $g$ is surjective.

6. Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4, 5, 6, 7\}$ and consider all possible functions $h : A \to B$.
   
   (a) How many such functions are there?
   
   (b) How many of those functions are injective?
   
   (c) How many of those functions are surjective?
   
   Explain your answers.

As always we should have at least a couple of harder problems on the homework. Here they are:

7. Prove that $f$ is bijective if and only if $f \circ f$ is bijective. Use this to show that the function $f : (0, \infty) \to (0, \infty)$ defined by $f(x) = \ln \left(\frac{e^x + 1}{e^x - 1}\right)$ is bijective.
   
   Hint: You may need to show that if $f \circ g$ is injective then $g$ is injective, and then show that if $f \circ g$ is surjective then $f$ is surjective

8. For $n \in \mathbb{N}$, let $A$ be a set with exactly $n$ elements and let $B$ be a set with exactly two elements. That is, let $A = \{a_1, a_2, a_3, \ldots, a_n\}$ (where all the $a_j$ are distinct) and $B = \{0, 1\}$. Now let $F$ be the set of all functions $f : A \to B$.
   
   (a) What is $|F|$, the size of $F$?
Now define a new function $g$ from the set $F$ to the power set of $A$, defined by

$$g : F \to \mathcal{P}(A) \quad \quad g(f) = \{a \in A : f(a) = 1\}$$

(b) Is the function $g$ injective? Prove your answer.

(c) Is the function $g$ surjective? Prove your answer.