Homework 5

(1) Chapter 10: Question 8

(2) Chapter 10: Question 18

(3) Chapter 10: Question 22

(4) The Fibonacci numbers are defined by the recurrence

\[ F_1 = 1 \quad F_2 = 1 \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \quad \text{for} \quad n > 2. \]

Show that for all \( k \in \mathbb{N} \), \( F_{4k} \) is a multiple of 3.

Here are the first few Fibonacci numbers:

\[ (F_n) = (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, \ldots) \]

(5) Let \( f(x) = x\ln x \) and \( x > 0 \). Let \( f^{(n)}(x) \) denote the \( n \)th derivative of \( f(x) \) for \( n \in \mathbb{N} \).

Prove that

\[ f^{(n)}(x) = (-1)^n \frac{(n-2)!}{x^{n-1}} \]

for all integers \( n \geq 3 \).

The next two are a slightly harder:

6. In cases where proving the inductive step is harder for a proof by induction, one can use another induction method, called strong induction, and it goes as follows:

**Theorem 1.** A statement of the form “\( \forall n \in \mathbb{N}, P(n) \)” is true if

- The statement \( P(1) \) is true,

  and,

- given \( k \geq 1 \), \( P(1) \land P(2) \land P(3) \land \ldots \land P(k) \) \( \implies \) \( P(k+1) \).

Use this result to prove the following statement:

Suppose you begin with a pile of \( n \) stones \( (n \geq 2) \) and split this pile into \( n \) separate piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have \( p \) and \( q \) stones in them, respectively, you compute \( pq \). Show that no matter how you split the piles (eventually into \( n \) piles of one stone each), the sum of the products computed at each step equals \( \frac{n(n-1)}{2} \).

For example — say with start with 5 stones and split them as follows:

\[ (5) \mapsto (2), (3) \mapsto (1), (1), (2), (1) \mapsto (1), (1), (1), (1) \]

So we total up \( 6 + 1 + 2 + 1 = 10 = \frac{5 \times 4}{2} \) \( \checkmark \)
7. This question involves two definitions:

- We say that a function, $f$, is “essentially effective”, if $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $y \geq x$ and $|f(y)| \geq 1$.

- We also say that a function, $g$, is “almost effective”, if $\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}$, $(y \geq x) \implies |f(y)| \geq 1$.

We’ve just made this up and are not actually used in mathematics, but we’ll just use them in this question.

Use these definitions to prove, or find counter-examples to disprove the following two statements:

(a) If $f$ is essentially effective, then it is almost effective.

(b) If $f$ is almost effective, then it is essentially effective.