The non-local mean-field equation on an interval

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Abstract

We consider the fractional mean-field equation on the interval $I = (-1, 1)$

$(-\Delta)^{\frac{1}{2}} u = \rho \frac{e^u}{\int_I e^u dx},$

subject to Dirichlet boundary conditions, and prove that existence holds if and only if $\rho < 2\pi$. This requires the study of blowing-up sequences of solutions. We provide a series of tools in particular which can be used (and extended) to higher-order mean field equations of non-local type.

1 Introduction

Given a number $\rho > 0$, we consider the non-local mean-field equation

$(-\Delta)^{\frac{1}{2}} u = \rho \frac{e^u}{\int_I e^u dx}, \quad I = (-1, 1)$ (1)

subject to the Dirichlet boundary condition

$u \equiv 0$ in $\mathbb{R} \setminus I.$ (2)

There are different ways to define the fractional Laplacian $(-\Delta)^{\frac{1}{2}}$ and therefore make sense of Problem (1)-(2). Consider the space of functions $L^1_2(\mathbb{R})$ defined by

$L^1_2(\mathbb{R}) = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}) : \int_{\mathbb{R}} \frac{|u(x)|}{1 + |x|^2} dx < \infty \right\}. (3)$

For a function $u \in L^1_2(\mathbb{R})$ one can define $(-\Delta)^{\frac{1}{2}} u$ as a tempered distribution as follows:

$\langle (-\Delta)^{\frac{1}{2}} u, \varphi \rangle := \int_{\mathbb{R}} u(-\Delta)^{\frac{1}{2}} \varphi dx, \quad \varphi \in \mathcal{S},$ (4)

where $\mathcal{S}$ denotes the Schwartz space of rapidly decreasing smooth functions and for $\varphi \in \mathcal{S}$ we set

$(-\Delta)^{\frac{1}{2}} \varphi := \mathcal{F}^{-1}(\cdot |\cdot \hat{\varphi}).$
Here the Fourier transform is defined by

\[ \hat{\phi}(\xi) \equiv \mathcal{F}(\phi)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} \phi(x) \, dx. \]

Notice that the convergence of the integral in (4) follows from the fact that for \( \phi \in \mathcal{S} \) one has

\[ |(-\Delta)^{1/2} \phi(x)| \leq C(1 + |x|^2)^{-1}. \]

If \( u \in C^{0,\alpha}(I) \) we can also define

\[ (-\Delta)^{1/2} u(x) := \frac{1}{\pi} \text{P.V.} \int_{\mathbb{R}} \frac{u(x) - u(y)}{(x - y)^2} \, dy, \quad x \in I. \]

These definitions are equivalent for the functions that we shall consider, namely function in \( C^{0,\frac{1}{2}}(\mathbb{R}) \) vanishing outside \( I \). In fact, every solution to (1)-(2) lies in \( C^{0,\frac{1}{2}}(\mathbb{R}) \cap C^\infty(I) \), see e.g. Corollary 1.6 of [25], and it is smooth inside \( I \) by a standard bootstrap argument. Therefore there is no loss of generality in working only with functions in \( C^{0,\frac{1}{2}}(\mathbb{R}) \cap C^\infty(I) \).

In this paper we shall develop some tools to treat existence and non-existence for problem (1)-(2). In spite of the possibility of working with the extension of \( u \) to the upper half-plane, i.e. of localizing the problem as often done, we will only use purely non-local methods, that can be best extended to treat also non-local higher-dimensional cases.

In dimension 2 the analog of Problem (1)-(2) is

\[ -\Delta u = \rho \frac{e^u}{\int_{\Omega} e^u \, dx} \quad \text{in} \, \Omega, \quad u = 0 \quad \text{on} \, \partial \Omega, \quad \Omega \subset \mathbb{R}^2 \]

where \( \Omega \) is smoothly bounded. As proven in [3] using variational arguments (minimization of a suitable functional) and in [15] via probabilistic methods, Problem (5) has a solution for every \( \rho \in (0, 8\pi) \). The threshold \( 8\pi \) is sharp since when \( \Omega \) is star-shaped (5) has no solution for every \( \rho \geq 8\pi \) by the Pohozaev identity.

If, on the other hand, \( \Omega \) is not simply connected or it is replaced by a closed Riemann surface \((\Sigma, g)\) of genus at least 1, in which case (5) is replaced by

\[ -\Delta_g u = \rho \left( \frac{e^u}{\int_{\Sigma} e^u \, dv_g} - 1 \right) \quad \text{in} \, \Sigma, \]

Ding-Jost-Li-Wang [8] proved that (6) admits a solution for every \( \rho \in (8\pi, 16\pi) \). Struwe and Tarantello [26] independently proved a similar result on the flat torus and for \( \rho \in (8\pi, 4\pi^2) \). For a general closed surface (including a sphere) Malchiodi [18] proved existence for every \( \rho \notin 8\pi N \), using the barycenter technique, see also [9].

An important tool in proving such existence results is an a priori study of the blowing-up behavior of sequences \((u_k)\) of solutions to (5) or (6) with \( \rho = \rho_k \). This was performed by Brezis-Merle [3] and Li-Shafrir [16] for the Liouville equation, which arises from (5) by adding a constant. Theses seminal works have several extensions to even dimension 4 and higher, see e.g. [28], [24] and [22], using higher-dimensional compactness results, see e.g. [19]. In order to study the 1-dimensional case we will need the following analogue non-local blow-up result.
**Theorem 1** Let \( u_k \) be a sequence of solutions to \((1), (2)\) with \( \rho = \rho_k > 0 \). Then up to a subsequence one of the following is true:

(i) \((u_k)\) is bounded in \( C^{0, \frac{1}{2}}(\mathbb{R}) \cap C^\ell_{\text{loc}}(I) \) for every \( \ell \in \mathbb{N} \).

(ii) \( \lim_{k \to \infty} u_k(0) = \infty \) as \( k \to \infty \).

Moreover, for \( 0 < \sigma < \frac{1}{2} \)

\[ u_k \to 2\pi G_0 \quad \text{in} \quad C^{0, \sigma}_{\text{loc}}(\mathbb{R} \setminus \{0\}), \]

where \( G_0 \) is the Green function of \((-\Delta)^\frac{1}{2}\) on \( I \) with Dirichlet boundary condition.

Let us notice that if we replace the right-hand side of \((1)\) with the nonlinearity \( e^{u^2} \), nonlocal compactness problems have been studied in [14] and [17], but the techniques used there are different, for instance because of the lack of a Pohozaev-type identity. In fact a result analog to \((8)\) is still unknown in the fractional case, although in dimension 2 it was recently proven by Druet-Thizy [10], see also [21].

Using Theorem 1 and Schauder’s fixed-point theorem we are able to prove the following result about existence and non-existence.

**Theorem 2** There exists a non-trivial non-negative solution \( u = u_\rho \) to \((1), (2)\) if and only if \( \rho \in (0, 2\pi) \). Moreover,

\[ u_\rho(0) \to \infty \quad \text{as} \quad \rho \uparrow 2\pi. \]

Although our method is topological, it is plausible that a variational argument in the spirit of [1] can also be employed.

The non-existence for \( \rho \geq 2\pi \) follows at once from a Pohozaev-type inequality (see Proposition 6), consistently with the non-existence in dimension 2 when the domain is star-shaper. Notice that the critical threshold \( 2\pi \) in Theorem 2 corresponds to the value \( 8\pi \) for Problems (5) and (6).

The last statement of Theorem 2 is about the existence of blowing-up sequences of solutions, namely it shows that the situation presented in Case (ii) of Theorem 1 actually occurs. The proof will follow by contradiction, together with the non-existence result of \( \rho = 2\pi \). In dimension 2 and higher, several such results (sometimes very subtle) are obtained by the Lyapunov-Schmidt reduction. For instance, when \( \Omega \) is simply connected, Weston [29] proved existence of solutions to \((5)\) blowing-up on a critical point of the Robin function of \( \Omega \), and [8] extended this result to the non-simply connected case. Multi-peak solutions have also been constructed, starting with the seminal work of Baraket-Pacard [1], see e.g. the work [11] and its references.

We also mention that in dimension 2, when \( \Omega \) is simply connected and \( \rho \in (0, 8\pi) \), Suzuki [27] proved uniqueness of solutions for Problems (5). It is reasonable to expect that the same holds in 1 dimension for (1)-(2).

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2 Preliminaries

We shall use the Green function defined by the formula
\[
G(x, y) := \frac{1}{\pi} \left( \log(\sqrt{1 - |x|^2})(1 - |y|^2) + 1 - xy \right) - \log|x - y| =: -\frac{1}{\pi} \log|x - y| + G(x, y), \quad x, y \in I
\]
and \(G(x, y) = 0\) for \(x \in I, y \in \mathbb{R} \setminus I\). It is well-known (see e.g. [2]) that
\[
(-\Delta)^\frac{1}{2} G_x = \delta_x \quad \text{for} \; x \in I.
\]

As usual, using the Green function we can write solutions to (1)-(2) in terms of a Green representation formula.

Lemma 3 A function \(u \in C^2(\mathbb{R}) \cap C^\infty(I)\) solves (1)-(2) if and only if
\[
u(x) = \rho \int_I G_x(y) \frac{e^{u(y)}}{\int_I e^{u(\xi)} d\xi} dy.
\]

Proof. This standard proof can be found for instance in the proof of [20, Proposition 7] (Identity (15) in particular).

Corollary 4 If \(u\) solves (1)-(2), then \(u > 0\) in \(I\).

In the following lemma we apply a non-local version of the famous moving-plane technique.

Lemma 5 Let \(u \in C^2(\mathbb{R}) \cap C^\infty(I)\) solve (1)-(2). Then \(u\) is even and decreasing, in the sense that \(u(x) = u(-x)\) and \(u(x) \geq u(y)\) for \(0 \leq x \leq y\).

Proof. This follows at once from the moving plane technique, see Theorem 11 in the Appendix.

Proposition 6 Let \(\tilde{u} \in C^1(\mathbb{R}) \cap C^\infty(I)\) be a solution to
\[
\tilde{u}(x) = \int_I G_x(y) e^{\tilde{u}(y)} dy + c,
\]
for some \(c \in \mathbb{R}\). Then for
\[
\rho := \int_I e^{\tilde{u}(y)} dy,
\]
we have \(\rho < 2\pi\).

Proof. We fix \(\psi \in C^1((0, \infty))\) such that \(\psi = 0\) on \([0, 1]\) and \(\psi = 1\) on \((2, \infty)\). Set for \(\varepsilon > 0\) small enough, \(\psi_\varepsilon(x) := \psi(\frac{x}{\varepsilon})\). We can rewrite \(\tilde{u}\) as
\[
\tilde{u}(x) = \frac{1}{\pi} \int_I \log\left( \frac{1}{|x - y|} \right) \psi_\varepsilon(|x - y|) e^{\tilde{u}(y)} dy + \int_I H(x, y) e^{\tilde{u}(y)} dy + w(x) + c,
\]
(11)
where
\[ w(x) := w_{\psi, \epsilon}(x) := \frac{1}{\pi} \int_I \log \left( \frac{1}{|x - y|} \right) \left( 1 - \psi_\epsilon(|x - y|) \right) e^{\tilde{u}(y)} dy. \] (12)

Note that by definition of \( \psi_\epsilon \) we integrate only on \([-2\epsilon, 2\epsilon]\), so we obtain
\[ \|w\|_{L^\infty(I)} \leq C\epsilon |\log \epsilon| \|e^{\tilde{u}}\|_{L^\infty(I)}. \]

Moreover, \( w \in C^1(I) \), which follows from \( \tilde{u} \in C^1(I) \). Differentiating under the integral sign in (11) we get
\[ \tilde{u}'(x) = \frac{1}{\pi} \int_I \frac{\partial}{\partial x} \left( \log \frac{1}{|x - y|} \psi_\epsilon(|x - y|) \right) e^{\tilde{u}(y)} dy + \int_I \frac{\partial}{\partial x} H(x, y) e^{\tilde{u}(y)} dy + w'(x). \]

We define \( I_1 \) as the quantity that we obtain multiplying the above identity by \( xe^{\tilde{u}(x)} \) and integrating over \( I \), i.e,
\[ I_1 := \int_I x\tilde{u}'(x)e^{\tilde{u}(x)} dx. \]

On the one hand, since \( \tilde{u} \) is even by Lemma 11 integration by parts yields
\[ I_1 = 2e^{\tilde{u}(1)} - \int_I e^{\tilde{u}} dx = 2e^{\tilde{u}(1)} - \rho. \] (13)

On the other hand, by definition
\[ I_1 = \frac{1}{\pi} \int_I \int_I x \frac{\partial}{\partial x} \left( \log \frac{1}{|x - y|} \psi_\epsilon(|x - y|) \right) e^{\tilde{u}(y)} e^{\tilde{u}(x)} dy dx 
+ \int_I \int_I x \frac{\partial}{\partial x} H(x, y) e^{\tilde{u}(y)} e^{\tilde{u}(x)} dy dx + \int_I w'(x) xe^{\tilde{u}(x)} dx 
=: I_2 + I_3 + I_4. \]

Using that \( \psi_\epsilon = 0 \) on \([0, \epsilon]\) we obtain
\[ I_2 = \frac{1}{\pi} \int_I \int_I x \left( -\psi_\epsilon(|x - y|) + \log \frac{1}{|x - y|} \psi_\epsilon'(|x - y|) \frac{x - y}{|x - y|} \right) e^{\tilde{u}(y)} e^{\tilde{u}(x)} dy dx 
= -\frac{1}{2\pi} \int_I \int_I \psi_\epsilon(|x - y|) e^{\tilde{u}(y)} e^{\tilde{u}(x)} dy dx - \frac{1}{2\pi} \int_I \int_I F(x, y) dy dx 
+ \frac{1}{2\pi} \int_I \int_I \log \frac{1}{|x - y|} |x - y| \psi_\epsilon'(|x - y|) e^{\tilde{u}(y)} e^{\tilde{u}(x)} dy dx 
=: J_1 + J_2 + J_3, \]

where
\[ F(x, y) := \frac{x + y}{x - y} \left( \psi_\epsilon(|x - y|) - \log \frac{1}{|x - y|} \psi_\epsilon'(|x - y|) |x - y| \right) e^{\tilde{u}(y)} e^{\tilde{u}(x)}. \]

By dominated convergence theorem, using the definition and regularity of \( \psi \) we can assert
\[ J_1 \xrightarrow{\epsilon \to 0} -\frac{\rho^2}{2\pi}; \quad J_3 \xrightarrow{\epsilon \to 0} 0. \]
Moreover, since \( F(x, y) = -F(y, x) \), we have \( J_2 = 0 \). Therefore, we get
\[
I_2 \xrightarrow{\varepsilon \to 0} -\frac{\rho^2}{2\pi}.
\] (14)

We claim now that \( I_3 < 0 \). To prove it, we first compute
\[
x \frac{\partial}{\partial x} H(x, y) = x \frac{1}{\pi} \frac{\partial}{\partial x} \log \left( \sqrt{(1 - x^2)(1 - y^2) + 1 - xy} \right)
= \frac{x}{\pi} \frac{-y - x \sqrt{1 - y^2}}{\sqrt{(1 - x^2)(1 - y^2) + 1 - xy}}
\leq \frac{x}{\pi} \frac{-y}{\sqrt{(1 - x^2)(1 - y^2) + 1 - xy}},
\]
This inequality together with Lemma 11 (which implies \( \hat{u}(-x) = \hat{u}(x) \)) proof the claim as follows
\[
I_3 \leq -\frac{1}{\pi} \int_0^1 \int_0^1 \frac{xy}{\sqrt{(1 - x^2)(1 - y^2) + 1 - xy}} e^{\hat{u}_k(y)} e^{\hat{u}_k(x)} dy dx
=: -\frac{2}{\pi} \int_0^1 \int_0^1 K(x, y) e^{\hat{u}_k(y)} e^{\hat{u}_k(x)} dy dx
< 0,
\]
where the last inequality follows from
\[
K(x, y) := xy \left( \frac{1}{\sqrt{(1 - x^2)(1 - y^2) + 1 - xy}} - \frac{1}{\sqrt{(1 - x^2)(1 - y^2) + 1 + xy}} \right) > 0
\]
on \((0, 1) \times (0, 1)\).

Finally we show that \( I_4 \to 0 \) as \( \varepsilon \to 0 \). Indeed, by integration by parts and using the bound for the function \( w \) defined in (12),
\[
I_4 = -\int_I (1 + \hat{u}') e^{\hat{u}} wd x + o_\varepsilon(1) = o_\varepsilon(1) + o_\varepsilon(1) \int_I |\hat{u}'| e^{\hat{u}} dx \xrightarrow{\varepsilon \to 0} 0,
\]
where we used that \( \hat{u}' e^{\hat{u}} \in L^1(I) \). Indeed, by Lemma 11 \( \hat{u}' \leq 0 \) on \((0, 1)\), so we have
\[
\int_0^1 |\hat{u}'| e^{\hat{u}} dx = \int_0^1 -(e^{\hat{u}})' dx = (e^{\hat{u}(0)} - 1) < \infty.
\]
Summarising, we obtain that
\[
I_1 = I_2 + I_3 + I_4 < I_2 + I_4 \xrightarrow{\varepsilon \to 0} -\frac{\rho^2}{2\pi},
\]
The proposition follows immediately from (13). □
3 Proof of Theorem

We set

\[ \hat{u}_k := u_k - \alpha_k, \quad \alpha_k := \log \left( \frac{\int_I e^{u_k} dx}{\rho_k} \right). \]  

(15)

Using Lemma 3 we write

\[ \hat{u}_k(x) = \int_I G_x(y) e^{\tilde{u}_k(y)} dy - \alpha_k, \quad \int_I e^{\tilde{u}_k} dx = \rho_k, \]  

(16)

and

\[ u_k(x) = \int_I G_x(y) e^{\tilde{u}_k(y)} dy. \]  

(17)

If \( \hat{u}_k(0) \leq C \), then by (17) \( u_k \) is bounded in \( C^0 \) \([-1,1]\) for every \( \alpha \in [0, \frac{1}{2}] \) and in \( C^\ell_{\text{loc}}\) \([-1,1]\) for \( \ell \geq 0 \), so that possibility (i) in the theorem occurs.

In the following we assume that \( \hat{u}_k(0) \to \infty \) and we shall set

\[ r_k := 2e^{-\hat{u}_k(0)} \to 0. \]

Lemma 7

Assume that \( \hat{u}_k(0) \to \infty \). Then we have

i) \( r_k u_k(0) \to 0 \).

ii) \( \eta_k(x) := \hat{u}_k (r_k x) + \log(r_k) \to \eta_0(x) := \log \frac{2}{1+x^2} \) in \( C^\infty_{\text{loc}}(\mathbb{R}) \).

iii) \( \lim_{R \to \infty} \lim_{k \to \infty} \int_{-R r_k}^{R r_k} e^{\tilde{u}_k} dx = 2\pi. \)

iv) \( \alpha_k \to \infty \).

v) \( \hat{u}_k \to -\infty \) in \( C_0^0(\bar{I} \setminus \{0\}) \).

Proof. Step 1 We show that \( r_k u_k(0) \to 0 \).

Indeed from (17), for every \( \delta > 0 \)

\[ u_k(0) = \left( \int_{|y| < \delta} + \int_{\delta < |y| < 1} \right) G_0(y) e^{\tilde{u}_k(y)} dy \]

\[ \leq e^{\hat{u}_k(0)} \int_{|y| < \delta} G_0(y) dy + e^{\tilde{u}_k(\delta)} \int_{\delta < |y| < 1} G_0(y) dy \]

\[ \leq C e^{\hat{u}_k(0)} \delta |\log \delta| + e^{\tilde{u}_k(\delta)}. \]

Note that for both inequalities we have used that, by Lemma 11, \( \hat{u} \) is decreasing on \( |y| \).

Since \( \delta > 0 \) is arbitrary small, and \( \hat{u}_k(\delta) \not\to \infty \) (otherwise Proposition 5 would be violated), multiplying both sides of the inequality by \( r_k \), letting \( k \to \infty \) and \( \delta \to 0 \) we complete the proof of i).

We will divide the proof of part ii) in three main steps:

Step 2 For every \( \varepsilon > 0 \) there exists \( R \gg 1 \) such that for \( k \) large

\[ \int_{|x| > R} \frac{|\eta_k(x)|}{1+x^2} dx < \varepsilon. \]
On the one hand, by definition of \( \eta_k \) and \( r_k \) and by \([17]\) (which implies \( u(x) = 0 \) if \( |x| > 1 \)) we obtain that \( \eta_k(x) = \log r_k - \alpha_k = \log 2 - u_k(0) \) for \( |x| > r_k^{-1} \). Then, we can assert that

\[
\int_{|x| > r_k^{-1}} \frac{|\eta_k(x)|}{1 + x^2} dx \leq C u_k(0) r_k \xrightarrow{k \to \infty} 0.
\]

On the other hand, again by definition of \( \eta_k \) and \( r_k \) and by \([17]\), for \( |x| < r_k^{-1} \) we have

\[
\eta_k(x) - \log 2 = u_k(r_k x) - u_k(0)
\]

\[
= \frac{1}{\pi} \int_I \log \frac{|y|}{r_k x - y} \ e^{u_k(y)} dy + \frac{1}{\pi} \int_I (H(r_k x, y) - H(0, y)) e^{u_k(y)} dy
\]

\[
=: f_k(x) + g_k(x).
\]

First, we bound the first integral as follows. Changing the variable \( y \mapsto r_k y \) we obtain

\[
f_k(x) = \int_{|y| < r_k^{-1}} \log \left( \frac{|y|}{|x|} \right) \ e^{u_k(y)} dy,
\]

and with Fubini’s theorem we bound

\[
\int_{I_R} \frac{|f_k(x)|}{1 + x^2} dx \leq C \int_{|y| < r_k^{-1}} e^{u_k(y)} \int_{I_R} \log \frac{|y|}{|x|} \ dx \ dy, \quad I_R := (-r_k^{-1}, r_k^{-1}) \setminus (-R, R).
\]

We claim that for \( R \) sufficiently large

\[
\int_{I_R} \frac{|f_k(x)|}{1 + x^2} dx < \varepsilon.
\]

By the previous bound \([18]\), this would follow immediately once we prove

\[
\int_{I_R} \log \frac{|y|}{|x|} \ dx \ x^2 < \varepsilon \quad \text{for } |y| \geq 1.
\]

Note that the inequality is trivial if \( |y| < 1 \). Splitting \( I_R \) into

\[
I_R = \bigcup_{i=1}^3 A_i, \quad A_1 := \{ |x| < \frac{|y|}{2} \} \cap I_R, \quad A_2 := \{ |x| \geq 2|y| \} \cap I_R, \quad A_3 := I_R \setminus (A_1 \cup A_2)
\]

we write

\[
\int_{I_R} \log \frac{|y|}{|x|} \ dx \ x^2 = \sum_{i=1}^{3} \int_{A_i} \log \frac{|y|}{|x|} \ dx \ x^2 =: (I_1) + (I_2) + (I_3).
\]

Using that \( |x - y| \approx |y| \) on \( A_1 \) one gets

\[
(I_1) \leq C \int_{|x| > R} \frac{dx}{1 + x^2} < \frac{\varepsilon}{4}.
\]

Since \( |x - y| \approx |x| \) on \( A_2 \)

\[
(I_2) \leq C \int_{|x| > R} \frac{\log |x|}{1 + x^2} dx < \frac{\varepsilon}{4} \quad \text{for } |y| \geq 1.
\]

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Finally, we have $|y| \approx |x|$ on $A_3$, and using the assumption $|y| \geq 1$, we get for $R$ large enough

$$(I_3) \leq C \int_{|x| \geq R} \frac{\log |x|}{1 + x^2} dx + C \min\left\{ \frac{1}{R^2}, \frac{1}{y^2} \right\} \int_{A_3} \log |x-y||dx$$

$$\leq \varepsilon + C \min\left\{ \frac{1}{R^2}, \frac{1}{y^2} \right\} |y| \log(1+|y|)$$

$$< \varepsilon.$$  

This proves (19).

Using that $|H(x,y)| \leq C + |\log(1-|x|)|$, one easily gets

$$\int_{I_R} \frac{|g_k(x)|}{1 + x^2} dx < \varepsilon \quad \text{for } R \gg 1.$$

Step 2 follows.

**Step 3** (Equicontinuity) For every $\varepsilon > 0$ and $R > 0$ there exists $\delta = \delta(\varepsilon, R) > 0$ such that

$$|\eta_k(x_1) - \eta_k(x_2)| < \varepsilon \quad \text{for } |x_1 - x_2| < \delta \text{ with } x_1, x_2 \in (-R, R).$$

We have

$$\eta_k(x_1) - \eta_k(x_2) = \frac{1}{\pi} \int_{|y| \leq M} \log \frac{|r_k x_2 - y|}{|r_k x_1 - y|} e^{\hat{\eta}_k(y)} dy + \frac{1}{\pi} \int_{|y| \leq M} (H(r_k x_1, y) - H(r_k x_2, y)) e^{\hat{\eta}_k(y)} dy$$

$$=: f_k(x_1, x_2) + g_k(x_1, x_2).$$

It is easy to see, using the continuity of $H$, that $|g_k(x_1, x_2)| < \varepsilon$ if $\delta > 0$ is sufficiently small. For $M \gg R$

$$|f_k(x_1, x_2)| \leq \frac{1}{\pi} \left( \int_{|y| \leq Mr_k} + \int_{Mr_k \leq |y| \leq 1} \right) \left| \log \frac{|r_k x_2 - y|}{|r_k x_1 - y|} \right| \left| e^{\hat{\eta}_k(y)} \right| dy$$

$$= \frac{1}{\pi} \int_{|y| \leq M} \left| \log \frac{|x_2 - y|}{|x_1 - y|} \right| e^{\eta_k(y)} dy + \frac{1}{\pi} \int_{M \leq |y| \leq r_k^{-1}} \left| \log \frac{|x_2 - y|}{|x_1 - y|} \right| e^{\eta_k(y)} dy$$

$$=: (I) + (II).$$

As $\eta_k \leq \log 2$, for every fixed $M > 0$ we can choose $\delta > 0$ so that $(I) < \varepsilon$. Since

$$\frac{|x_2 - y|}{|x_1 - y|} = 1 + |x_1 - x_2|O\left( \frac{1}{M} \right) \quad \text{for } x_1, x_2 \in (-R, R) \text{ and } |y| \geq M >> R,$$

one gets

$$(II) \leq \frac{C}{M} |x_1 - x_2| \int_{|y| \leq r_k^{-1}} e^{\eta_k(y)} dy \leq \frac{C}{M} |x_1 - x_2| \leq \frac{C \delta}{M}.$$  

This proves Step 3.

**Step 4** (Up to a subsequence) $\eta_k \to \eta$ in $C^0_{\text{loc}}(\mathbb{R})$ where $\eta$ satisfies ($S(\mathbb{R})$ is the Schwartz space)

$$\int_{\mathbb{R}} \eta(-\Delta)^{\frac{1}{2}} \varphi dx = \int_{\mathbb{R}} \varphi dx \quad \text{for every } \varphi \in \mathcal{S}(\mathbb{R}).$$
This follows directly, by Ascoli-Arzelà Theorem, from Steps 2 and 3. Then by a classification results, see e.g. [6, Theorem 1.8] or [7, Theorem 1.7], \( \eta = \eta_0 \), and \( \text{ii} \) is proven.

Moreover, as a corollary of \( \text{ii} \) we obtain \( \text{iii} \):

\[
\lim_{R \to \infty} \lim_{k \to \infty} \int_{-R}^{R} e^{\hat{u}_k} \, dx = \lim_{R \to \infty} \int_{-R}^{R} \frac{2}{1 + x^2} \, dx = 2\pi.
\]

**Step 5** We prove here \( \alpha_k \to \infty \).

Assume by contradiction that \( \alpha_k \not\to \infty \). Then for every \( \varepsilon > 0 \) and for \( k \) large, from \( \text{(16)} \), and together with \( \text{ii} \)

\[
\hat{u}_k(x) \geq \frac{1}{\pi} \int_{I} \log \left( \frac{1}{|x-y|} \right) e^{\hat{u}_k(y)} \, dy - C \geq \frac{3}{2} \log \frac{1}{|x|} - C, \quad \varepsilon \leq |x| \leq 1.
\]

This contradicts \( \rho_k < 2\pi \), thanks to Proposition 6. Thus, part \( \text{iv} \) is proved.

**Step 6** \( \hat{u}_k \to -\infty \) in \( C^0_{\text{loc}}(\overline{I} \setminus \{0\}) \).

Since \( \hat{u}_k \) is monotone decreasing, it is sufficient to show that \( \hat{u}_k(\varepsilon) \to -\infty \) for every \( \varepsilon > 0 \). As \( \hat{u}_k(\frac{x}{\varepsilon}) \not\to \infty \), which follows from \( \rho_k < 2\pi \) and the monotonicity of \( \hat{u}_k \), we have

\[
\hat{u}_k(\varepsilon) + \alpha_k \leq C(1 + |\log \varepsilon|) + \frac{1}{\pi} \int_{|x-\varepsilon| < \frac{x}{2}} \log \left( \frac{1}{|\varepsilon - y|} \right) e^{\hat{u}_k(y)} \, dy
\]

\[
\leq C(1 + |\log \varepsilon|) + Ce^{\hat{u}_k(\frac{x}{\varepsilon})}
\]

\[
\leq C(\varepsilon).
\]

This bound, together with Step 5, implies Step 6. In this way, we have proved part \( \text{v} \), and with it, the whole Lemma.

**Lemma 8** For \( \sigma \in (0, \frac{1}{2}) \) we have

\[
u_k \to 2\pi G_0 \text{ in } C^0_{\text{loc}}(\overline{I} \setminus \{0\}) \] (20)

**Proof.** \( C^0 \) convergence: We write

\[
u_k(x) - 2\pi G_0(x) = \int_{I} (G_x(y) - G_0(x)) e^{\hat{u}_k(y)} \, dy + (\rho_k - 2\pi)G_0(x)
\]

\[=: v_k(x) + (\rho_k - 2\pi)G_0(x).
\]

It follows that \( (\rho_k - 2\pi)G_0 \to 0 \) in \( C^\infty(\overline{I}) \), thanks to Proposition 6 and part \( \text{iii} \) of Lemma 7. For \( 0 < \varepsilon \leq |x| \leq 1 \) we have

\[
|G_x(y) - G_0(x)| \ll |y| \ll \varepsilon
\]

and

\[
|G_x(y) - G_0(x)| \leq C + C|x-y| + C|\log(1-|y|)| \text{ for } |y| < 1.
\]

This bound together with part \( \text{v} \) of Lemma 7 would imply \( v_k \to 0 \) in \( C^0_{\text{loc}}(\overline{I} \setminus \{0\}) \).

We claim that

\[
[u_k]_{C^0_{\text{loc}}((\varepsilon,1))] \leq C(\varepsilon) \text{ for every } \varepsilon > 0.
\]
Then the $C^{0,\sigma}_{\text{loc}}(\bar{I} \setminus \{0\})$ convergence for $\sigma < \frac{1}{2}$ will follow immediately from the Ascoli-Arzelà Theorem.

For $x \in (\varepsilon, 1)$ and $h > 0$ with $x + h \leq 1$ we have

$$u_k(x+h) - u_k(x) = \frac{1}{\pi} \int_{I} \log \frac{|x-y|}{|x+h-y|} e^{\hat{u}_k(y)} dy + \int_{I} (H(x+h,y) - H(x,y)) e^{\hat{u}_k(y)} dy$$

$$= I_1 + I_2$$

Since

$$\log \frac{|x-y|}{|x+h-y|} = O(h) \quad \text{for } |y| \leq \frac{\varepsilon}{2}, \quad x \geq \varepsilon, \quad h > 0,$$

and $\hat{u}_k \to -\infty$ in $C^{0,\sigma}_{\text{loc}}(\bar{I} \setminus \{0\})$ by part $v)$ in Lemma 7, we obtain

$$|I_1| \leq C_\varepsilon h + C \int_{I} \log \frac{|x-y|}{|x+h-y|} dy \leq C_\varepsilon h |\log h|.$$ 

In order to bound $I_2$ we use

$$H(x+h,y) - H(x,y)$$

$$= \int_{0}^{1} \frac{\partial}{\partial t} H(x+th,y) dt$$

$$= h \int_{0}^{1} \frac{-y - (x+th) \sqrt{1-y^2}}{(1-(x+th)^2)(1-y^2) + 1 - (x+th)y} dt$$

$$= O(h) \frac{1}{\sqrt{1-y^2}} \int_{0}^{1} \frac{dt}{\sqrt{1 - (x+th)y}} + O(h) \int_{0}^{1} \frac{1}{1 - (x+th)y} \frac{dt}{\sqrt{1 - (x+th)^2}}$$

$$= O(h) \frac{1}{\sqrt{1-y^2}} \int_{0}^{1} \frac{dt}{\sqrt{1 - (x+th)}}$$

$$= O(h) \frac{1}{\sqrt{1-y^2}} \int_{0}^{1} \frac{dt}{\sqrt{1 - (x+th)}}$$

$$= O(1) \frac{1}{\sqrt{1-y^2}} \left( \sqrt{1-x} - \sqrt{1-x-h} \right)$$

$$= O(\sqrt{h}) \frac{1}{\sqrt{1-y^2}},$$

(21)

where, since $x + th \leq 1 \quad \forall t \in (0,1)$, 2nd to 3rd equality follows from

$$\frac{1}{\sqrt{(1-(x+th)^2)}(1-y^2) + 1 - (x+th)y} \leq \min \left\{ \frac{1}{\sqrt{(1-(x+th)^2)(1-y^2)}}, \frac{1}{1 - (x+th)y} \right\}$$

and 3rd to 4th equality follows from

$$\frac{\sqrt{1-y^2}}{1 - (x+th)y} \leq \frac{1}{1 - |y|} \leq C \frac{1}{\sqrt{1-y^2}}.$$ 

Therefore

$$|I_2| \leq C \sqrt{h} \int_{I} \frac{1}{\sqrt{1 - |y|}} e^{\hat{u}_k(y)} dy \leq C \sqrt{h}.$$ 

This proves our claim. \qed
4 Proof of Theorem 2

We set
\[ X := C^0([-1, 1]), \quad \|u\|_X := \max_{[-1, 1]} |u(x)|. \]

We define \( T_\rho : X \to X \) given by
\[ T_\rho(u)(x) := \rho \int_I G(x, y) \frac{e^{u(y)}}{\int_I e^{u(\xi)} d\xi} dy. \]

Lemma 9 For every \( \rho > 0 \) the operator \( T_\rho \) is compact.

Proof. Let \( (u_k) \) be a sequence of functions in \( X \) such that \( \|u_k\|_X \leq M \). Then, up to a subsequence,
\[ \int_I e^{u_k} dx \to c_0, \]
for some \( c_0 > 0 \). Moreover, there exists \( C = C(M, \rho) > 0 \) such that for every \( x_1, x_2 \in I \)
\[ |T_\rho(u_k)(x_1) - T_\rho(u_k)(x_2)| \leq C \int_I \log \left| \frac{x_1 - y}{x_2 - y} \right| dy + C \int_I |H(x_1, y) - H(x_2, y)| dy \]
\[ \leq C|x_1 - x_2|^\frac{1}{2}, \]
where we have used that
\[ |H(x_1, y) - H(x_2, y)| \leq C \sqrt{|x_1 - x_2|} (1 - |y|)^\frac{1}{2}, \]
which follows from (21). Thus, the sequence \( (T_\rho(u_k)) \) is bounded in \( C^1(I) \), and hence, it is pre-compact in \( X \).

Proof of Theorem 2 (completed). Non-existence of solutions to (1)-(2) for \( \rho \geq 2\pi \) follows at once from Proposition 6.

We will use the Schauder fixed-point theorem to prove that \( T_\rho \) has a fixed point (say) \( u_\rho \) for every \( \rho \in (0, 2\pi) \), which by Lemma 3 will be a solution to (1)-(2). Fix \( \rho \in (0, 2\pi) \), and consider any sequence \( (t_k, u_k) \in (0, 1] \times X \) such that \( u_k = t_k T_\rho(u_k) \). Then \( u_k \) satisfies (1)-(2) with \( \rho \) replaced by \( \rho t_k < 2\pi \). Therefore, by Theorem 1 there exists \( C > 0 \) such that \( \|u_k\|_X \leq C \). Hence, by Schauder’s theorem, \( T_\rho \) has a fixed point in \( X \), which is a solution to (1)-(2).

For \( \rho \in (0, 2\pi) \) let \( u_\rho \in X \) be a fixed point of \( T_\rho \). Since \( T_{2\pi} \) does not have a fixed point, thanks to Proposition 6 and, since \( u_\rho(0) = \max_I u(\rho) \) by Lemma 1, we must have
\[ u_\rho(0) \to \infty \quad \text{as} \quad \rho \to 2\pi. \]
5 Appendix

We present here a self-contained proof of the non-local moving-plane technique in the simple case of an interval. It will be based on the following non-local Hopf-type lemma, which is now a rather classical result (see e.g. [5, Theorem 1], [12, Lemma 1.2] or [13, Lemma 2.7]). Since we could not find a reference fitting our assumptions, for the convenience of the reader, we present a proof here.

Lemma 10 (Hopf-type lemma) Let \( w \in L^\infty(\mathbb{R}) \cap C^0(\mathbb{R}) \) be a solution to

\[
\begin{cases}
(\Delta)^{\frac{1}{2}} w(x) = c(x)w(x) & \text{on } (a, 0) \\
w(x) = -w(-x) & \text{on } \mathbb{R} \\
w \leq 0 & \text{on } (-\infty, 0)
\end{cases}
\]

for some bounded function \( c \), and \( a \in [-\infty, 0) \). Assume that \( w \) is \( C^3 \) in a neighborhood of the origin. Then \( w \equiv 0 \) on \( \mathbb{R} \) if and only if \( w'(0) = 0 \).

Proof. We assume by contradiction that \( w \not\equiv 0 \) on \( \mathbb{R} \) and \( w'(0) = 0 \). Then, as \( w \) is an odd function, we have \( w(0) = w'(0) = w''(0) = 0 \). Hence, by Taylor expansion, for some \( \delta > 0 \)

\[
w(y) - w(x) = (y - x)w'(x) + \frac{(y-x)^2}{2}w''(x) + O((x - y)^3)
\]

\( w(x) = O(x^3), \quad w'(x) = O(x^2), \quad w''(x) = O(x) \),

(22)

for every \( x, y \in (-\delta, \delta) \). For \( x < 0 \) near the origin we write

\[
(-\Delta)^{\frac{1}{2}} w(x) = \frac{1}{\pi} \text{PV} \int_{\mathbb{R}} \frac{w(x) - w(y)}{(x - y)^2} dy
\]

\[
= \frac{1}{\pi} \text{PV} \left( \int_{y < 0} K(x, y)(w(x) - w(y)) dy + 2 \int_{y < 0} \frac{w(x)}{(x + y)^2} dy \right)
\]

\[
=: \frac{1}{\pi} \text{PV} \left( [I] + (II) \right),
\]

where

\[
K(x, y) := \left( \frac{1}{(x - y)^2} - \frac{1}{(x + y)^2} \right) > 0 \quad \text{on } (-\infty, 0) \times (-\infty, 0).
\]

This shows that \( w < 0 \) on \( (a, 0) \). Consequently, \( w \leq -M \) on \( (a_1, a_2) \) for some \( M > 0 \), where \( a < a_1 < a_2 < 0 \). For \( x < 0 \) very close to the origin and for \( |a_2| >> \varepsilon >> |x| \) we split \( (-\infty, 0) \) into \( (-\infty, 0) = \bigcup_{i=1}^{5} A_i \) where

\[
A_1 := (2x, 0), \quad A_2 := (-\varepsilon, 2x), \quad A_3 := (a_1, a_2), \quad A_4 := (a_2, -\varepsilon), \quad A_5 := (-\infty, a_1).
\]

We now write

\[
[I] = \sum_{i=1}^{5} I_i, \quad I_i := \int_{A_i} K(x, y)(w(x) - w(y)) dy.
\]

Using (22) we obtain

\[
\int_{A_1} \frac{w(x) - w(y)}{(x + y)^2} dy = O(x^2).
\]
Therefore, as \( I_1 \) is in the PV sense, again by (22)
\[
I_1 = O(x^2) + \text{PV} \int_{2x}^0 \frac{w'(x) + \frac{1}{2}(x - y)w''(x) + O((x - y)^2)}{x - y} dy = O(x^2).
\]

It follows that
\[
K(x,y)|x-y|^3 \leq 4|x| \quad \text{and} \quad K(x,y) \leq \frac{2}{(x-y)^2} \quad \text{for } y \in A_2.
\]

Hence, together with (22) one gets
\[
I_2 = O(\varepsilon)|x|.
\]

Since \( K(x,y) \approx |x| \) and \( w(x) - w(y) \geq \frac{M}{2} \) for \( y \in A_3 \), we obtain
\[
I_3 \geq c_1|x| \quad \text{for some } c_1 > 0.
\]

Now we fix \( \varepsilon > 0 \) small enough so that
\[
|I_1| + |I_2| \leq \frac{1}{4} c_1|x|.
\]

Then, for \( \varepsilon >> -x > 0 \) we have \( w(x) - w(y) > 0 \) for \( y \in A_4 \), which leads to \( I_4 > 0 \).

Recalling that \( w \leq 0 \) on \((-\infty, 0)\), we have \( w(x) - w(y) \geq w(x) = O(x^3) \) for \( y \in A_5 \), which gives \( I_5 \geq O(x^4) \). Thus
\[
(I) \geq \frac{3}{4} c_1|x| + O(x^4).
\]

Note that
\[
(II) = O(x^2).
\]

Combining these estimates we obtain
\[
0 = (-\Delta)^{\frac{1}{2}} w(x) + c(x)w(x) \geq \frac{3}{4} c_1|x| + O(x^2) + c(x)O(x^3) = \frac{3}{4} c_1|x| + O(x^2) > 0,
\]

for \( x < 0 \) sufficiently small, a contradiction. \( \square \)

**Theorem 11** Let \( u \in C^1_\ast(R) \cap C^\infty(I) \) be a solution to

\[
\begin{cases}
(-\Delta)^{\frac{1}{2}} u = f(u) & \text{in } I \\
u = 0 & \text{in } \mathbb{R} \setminus I \\
u > 0 & \text{in } I,
\end{cases}
\]

where \( f \) is Lipschitz continuous, non-negative and non-decreasing. Then \( u \) is even and \( u(x) \geq u(y) \) for \( 0 \leq x \leq y \).

**Proof.** First, we claim that \( u \) is monotone decreasing on \((1-\varepsilon, 1)\) for some \( \varepsilon > 0 \). Although this follows from Lemma 1.2 in [12], we shall give a simple self-contained proof. We write
\[
u(x) = \frac{1}{\pi} v(x) + w(x),
\]
where
\[ v(x) := \int_I \log \left( \frac{1}{|x-y|} \right) f(u(y)) dy, \quad w(x) := \int_I H(x,y) f(u(y)) dy, \]
where \( H(x,y) \) is as in \( \text{[8]} \). Differentiating under the integral sign one obtains \( u' \leq C \) on \((0,1)\). For \( h \) small we have
\[ v(x+h) - v(x) = f(u(x)) \int_I \log \left( \frac{|x-y|}{|x+h-y|} \right) dy + \int_I \log \left( \frac{|x-y|}{|x+h-y|} \right) (f(u(y)) - f(u(x))) dy =: v_1(x,h) + v_2(x,h). \]
Using that \( u \in C^1_\frac{1}{2}(\mathbb{R}) \) one gets
\[ \lim_{h \to 0} \frac{v_2(x,h)}{h} = O(1) \quad \text{on } I. \]
Computing the integral explicitly we obtain
\[ \lim_{h \to 0} \frac{v_1(x,h)}{h} = f(u(x)) (\log(1-x) - \log(1+x)) \quad \text{on } I. \]
Thus, for \( \varepsilon > 0 \) sufficiently small
\[ u'(x) \leq C + \frac{1}{\pi} f(u(x)) (\log(1-x) - \log(1+x)) < 0 \quad \text{on } (1-\varepsilon,1), \]
proving the claimed monotonicity. In particular, as \( u = 0 \) on \( I^c \) and \( u > 0 \) on \( I \), for \( \lambda > 1 - \frac{\varepsilon}{2} \) we have
\[ u_\lambda(x) := u(x) - u(x) \leq 0 \quad \text{on } \Sigma_\lambda := (-\infty, \lambda), \quad x_\lambda := 2\lambda - x. \]
We set
\[ \lambda^* := \inf \{ \lambda > 0 : u_\lambda \leq 0 \text{ on } \Sigma_\lambda \text{ for every } \lambda \geq \lambda \}. \]
We claim now that \( \lambda^* = 0 \). Otherwise there would be a sequence \( \lambda_n \uparrow \lambda^* > 0 \) and \( x_n \in \Sigma_{\lambda_n} \) such that
\[ \max_{\Sigma_{\lambda_n}} u_{\lambda_n} = u_{\lambda_n}(x_n) > 0. \]
Moreover, since \( u(x) = 0 \) for \( x \geq 1 \) and \( u > 0 \) in \( I \), we must have \( x_n \in (-1 + 2\lambda_n, \lambda_n) \). Then, up to a subsequence, \( x_n \to x_0 \in [-1 + 2\lambda^*, \lambda^*] \) and \( u_{\lambda^*}(x_0) = 0 \). Now, on the one hand, using the equation we have
\[ (-\Delta)^{\frac{1}{2}} u_{\lambda^*}(x) = f(u(x)) - f(u(x)) \leq 0 \quad \text{for } x \in (-1 + 2\lambda^*, \lambda^*). \]
On the other hand, with the singular kernel definition for the fractional Laplacian, since \( u_{\lambda^*} \leq 0 \) on \( \Sigma_{\lambda^*} \), \( u_{\lambda^*}(x_0) = 0 \) and \( u_{\lambda^*}(x) = -u_{\lambda^*}(x_{\lambda^*}) \), we can compute its value at \( x_0 \in [-1 + 2\lambda^*, \lambda^*] \):
\[ (-\Delta)^{\frac{1}{2}} u_{\lambda^*}(x_0) = \frac{1}{\pi} P.V. \int_{\mathbb{R}} \frac{u_{\lambda^*}(x_0) - u_{\lambda^*}(y)}{(x_0 - y)^2} dy \]
\[ = \frac{1}{\pi} P.V. \int_{\Sigma_{\lambda^*}} u_{\lambda^*}(y) \left( \frac{1}{(x_0 - 2\lambda^* - y)^2} - \frac{1}{(x_0 - y)^2} \right) dy \]
\[ \geq 0. \]
Then, we conclude that \( x_0 = \lambda^* \). Hence,
\[
0 = u_{\lambda^*}'(x_0) = -2u'(\lambda^*).
\]
Moreover, \( u_{\lambda^*} < 0 \) in \((-1 + 2\lambda^*, \lambda^*)\), and this contradicts Lemma 10. Thus \( \lambda^* = 0 \) and \( u_0 \leq 0 \). In a similar way one can show that \( u_0 \geq 0 \). □

References


