MSc Thesis

# On the topology of the exceptional Lie group $G_2$

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# Contents

# Chapter 1

# Introduction

This chapter presents the motivations behind the study of the exceptional Lie group $G_2$ and sets the aim of the research. Finally the structure of the thesis is briefly summarized.

## 1.1 Problem setting and background motivation

Lie groups form a central subject of modern mathematics and theoretical physics. They represent the best-developed theory of continuous symmetry of mathematical objects and structures, and this makes them neccesary tools in many parts of mathematics and physics. They provide a natural framework for analysing the continuous symmetries of structures, in much the same way as permutation groups are used in Galois theory for analysing the discrete symmetries of algebraic equations.

Lie groups are closely reated to Lie algebras, which can be thought as the study of infinitesimal transformations. They appear as linearizations of Lie groups around the identity element, and due to the special properties of Lie groups, several useful information about them can be recovered from these linear algebraic objects. By the celebrated result of Lie and Cartan there is a one-to-one correspondence between connected, simply-connected Lie groups and their Lie algebras.

The main question of the research is stated as follows. *What are the basic algebraic and topological properties of the group $G_2$ and in which areas of mathematics can this group be used?*

For addressing this problem, the following subquestions are specified:

1. What are the key concepts and results in the structure theory of Lie groups?

2. How can the Lie group $G_2$ be constructed?

3. For which other topological spaces can the Lie group $G_2$ provide some useful information?

## 1.2 The structure of the thesis

Because of the many different branches of research and vast number of interesting results in the topic it is not possible to cover everything in this short paper. We have focused rather on the most important concepts and the logical relations of the results. The rest of this thesis is organized as follows:

- **Chapter 2** summarizes the basic theory of Lie groups and principal bundles. We tried to collect the most important concepts and theorems without going into the technical details. Many theorems are presented without proofs, or with only an outline of the proof. Instead, the focus is rather on showing the importance of the different notions and results, and the relations between them.
- **Chapter 3** focuses on the algebraic structures that are necessary for constructing $G_2$ as the automorphism group of the Cayley algebra. As a well-known lower dimensional analogue the quaternion algebra and the group of its automorphisms are briefly presented.
- The details about the group $G_2$ are investigated in **Chapter 4**. First it is constructed as the automorphism group of the Cayley numbers. Then the transition functions are derived when $G_2$ is a principal bundle over $S^6$. As a corollary a new method is given to compute the generator of the group $\pi_5(SU(3))$. Finally $G_2$ is presented as the stabilizer of a specific 3-form with a short preview on $G_2$-manifolds.

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# Chapter 2

# Basic concepts

This chapter presents the most important theoretical concepts and results about Lie groups, Lie algebras and principal bundles. The notations and the underlying concepts are usually from differential geometry and topology. The material of this chapter can be found in graduate texts, for a detailed reference on these topics we refer to [10], [14], [15], and [17].

## 2.1 Smooth manifolds and maps

Let $M$ be a second countable Hausdorff topological space.

**Definition 2.1.** A *coordinate chart* (or just *chart*) on $M$ is a pair $(U, \varphi)$, where $U$ is an open subset of $M$ and $\varphi\colon U \to V$ is a homeomorphism from $U$ to an open subset $V = \varphi(U) \subset \mathbb{R}^n$. $U$ is then called a *coordinate neighborhood* of each of its points.

**Definition 2.2.** A $C^\infty$ *differentiable* or *smooth structure* on $M$ is a collection of coordinate charts $\{(U_\alpha, \varphi_\alpha)\}$ , where $\varphi_\alpha\colon U_\alpha \to V_\alpha \subseteq \mathbb{R}^n$, such that

1. $M = \cup_\alpha U_\alpha$.
2. Any two charts are smoothly compatible. That is, for every $\alpha$, $\beta$ the change of local coordinates $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a smooth ($C^\infty$) map on its domain of definition, i.e. on $\varphi_\alpha(U_\alpha \cap U_\beta)$.
3. The collection of charts $\varphi_\alpha$ is maximal with respect to the property 2: if a chart $\varphi$ of $M$ is compatible with all $\varphi$ then $\varphi$ is included in the collection.

Structures satisfying property 1 and 2 are called *atlases.* Smooth structures are precisley the maximal atlases and if an atlas is not maximal, we can take the maximal one containing it.

**Definition 2.3.** A *smooth manifold* is a pair $(M, \mathcal{A})$, where $M$ is a second countable Hausdoff topological space, and $\mathcal{A}$ is a smooth structure on it. Then $n$ is the *dimension* of $M$.

**Definition 2.4.** Let $M$, $N$ be smooth manifolds. A continuous map $f\colon M \to N$ is called a *smooth map* if for each $p \in M$ , for some (hence for every) charts $\varphi$ and $\psi$, of $M$ and $N$ respectively, with $p$ in the domain of $\varphi$ and $f(p)$ in the domain of $\psi$, the composition $\psi \circ f \circ \varphi^{-1}$ (which is a map between open sets in $\mathbb{R}^n$ , $\mathbb{R}^k$ , where $n = \dim M$, $k = \dim N$) is smooth on its domain of definition.

**Definition 2.5.** Two manifolds $M$ and $N$ are called *diffeomorphic* is there exists a smooth bijective map $M \to N$ having smooth inverse.

**Example 2.6.** Let $M$ be a smooth $n$-manifold and let $U \subset M$ be any open subset. Define an atlas on $U$ by $\mathcal{A}_U = \{\text{smooth charts } (V, \varphi) \text{ for } M \text{ such that } V \subset U\}$. Endowed with this smooth structure, we call any open subset an *open submanifold* of $M$.

### 2.1.1 Vector bundles

Let $M$ be a smooth manifold, $E$ be a connected Hausdorff space and $\pi\colon E \to M$ be a continous mapping.

**Definition 2.7.** The pair $(E, \pi)$ is a smooth (real) *vector bundle* of rank $k$ over $M$ if:

1. For each $p \in M$, the set $E_p = \pi^{-1}(p) \subset E$ (called the *fibre* of E over $p$) is endowed with the structure of a $k$-dimensional vector space.

2. For each $p \in M$, there exists a neighborhood $U$ of $p$ in $M$ and a homeomorphism $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ (called a *local trivialization* of $E$ over $U$), such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\ \Phi\ } & U \times \mathbb{R}^k \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi_1} & \\ & U & \end{array}$$

   where $\pi_1$ is the projection on the first component.

3. If $(U_\alpha, \Phi_\alpha)$ and $(U_\beta, \Phi_\beta)$ are two local trivializations of $E$, where $U_\alpha \cap U_\beta \neq \varnothing$, then for each $x \in U_\alpha \cap U_\beta$ the map $\Phi_{\alpha\beta}(x) = \Phi_\beta \circ \Phi_\alpha^{-1}(x)\colon \{x\} \times \mathbb{R}^k \to \{x\} \times \mathbb{R}^k$ is a linear mapping, which depends smoothly on $x$.

Condition 3 in the definition means that for each $\alpha$ and $\beta$, there exists a smooth function $\tau_{\alpha\beta}\colon U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$, such that $\Phi_{\alpha\beta}(x, v) = \Phi_\beta \circ \Phi_\alpha^{-1}(x, v) = (x, \tau_{\alpha\beta}(x)v)$. It is not too difficult to see that transition functions $\tau_{\alpha\beta}$ satisfy the following relations, called *cocycle conditions*

$$\begin{aligned} \tau_{\alpha\alpha} &= Id_{\mathbb{R}^k} \ , \\ \tau_{\beta\alpha}\tau_{\alpha\beta} &= Id_{\mathbb{R}^k} \ , \\ \tau_{\gamma\alpha}\tau_{\beta\gamma}\tau_{\alpha\beta} &= Id_{\mathbb{R}^k} \ . \end{aligned} \tag{2.1}$$

If $(E, \pi)$ is a vector bundle over $M$, then $E$ is called the *total space*, $M$ is called the *base space*, $\pi$ is the *projection mapping* and for each $p \in M$, the vector space $E_p = \pi^{-1}(p)$ is the *fibre* over $p$.

**Proposition 2.8.** *The total space $E$ of a vector bundle over a smooth manifold $M$ is a smooth manifold.*

**Definition 2.9.** A *section* of a vector bundle $\pi\colon E \to M$ is a continous function $\sigma\colon M \to E$, such that $\pi \circ \sigma = \mathrm{Id}_M$. The section is called smooth, if it is a smooth as a mapping between smooth manifolds. Specifically, this means that $\sigma(p)$ is an element of $E_p$ for each $p \in M$.

### 2.1.2 The tangent bundle

**Definition 2.10.** A linear map $X\colon C^\infty(M) \to M$ is called a *derivation at* $p$ if it satisfies the Leibniz rule: $X(fg) = f(p)Xg + g(p)Xf$ for all $f, g \in C^\infty(M)$. The set of derivations of $C^\infty$ at $p$ called the *tangent space* to $M$ at $p$, and is denoted by $T_pM$. An element of $T_pM$ is called a *tangent vector* at $p$.

It is easy to check that linear combination of derivations at $p$ is again a derivation, therefore the tangent space is a vector space.

**Example 2.11.** If $M = \mathbb{R}^n$ is covered by itself, then the derivations

$$\left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p$$

defined by

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \frac{\partial f}{\partial x^i}(p)$$

form a basis for $T_p(\mathbb{R}^n)$, which therefore has dimension $n$.

**Definition 2.12.** If $M$ and $N$ are smooth manifolds and $F\colon M \to N$ is a smooth map, for each $p \in M$ we define a linear map $F_*\colon T_pM \to T_{F(p)}N$, called the *pushforward* associated with $F$ by $(F_*X)(f) = X(f \circ F)$ for each $f \in C^\infty(N)$.

**Lemma 2.13** (Properties of Pushforwards)**.** *Let $F\colon M \to N$ and $G\colon N \to P$ be smooth maps, and let $p \in M$.*

*(a)* $(G \circ F) = G_* \circ F_*$.

*(b)* $(Id_M)_* = Id_{T_pM}$.

*(c) If $F$ is a diffeomorphism, then $F_*\colon T_pM \to T_{F(p)}N$ is an isomorphism.*

The next proposition shows that the tangent space is a purely local construction.

**Proposition 2.14.** *Suppose $M$ is a smooth manifold, $p \in M$ and $X \in T_pM$. If $f$ and $g$ are smooth functions on $M$ that agree on some neighbourhood of $p$, then $Xf = Xg$.*

**Proposition 2.15.** *Let $M$ be a smooth manifold, $U \subset M$ be an open submanifold and let $\iota\colon U \hookrightarrow M$ be the inclusion map. For any $p \in U$, $\iota_*\colon T_pU \to T_pM$ is an isomorphism.*

This means that if $U$ is an open set in a manifold $M$, then using the isomorphism $\iota_*\colon T_pU \to T_pM$, we can canonically identify $T_pU$ with $T_pM$. As a special case, let $(U, \varphi)$ be a coordinate chart on $M$. Then $\varphi$ is a diffeomorphism from $U$ to an open subset $\varphi(U) \subset \mathbb{R}^n$. By combining the results of Lemma 2.13 and Proposition 2.15, we see that $\varphi_*\colon T_pM \to T_{\varphi(p)}\mathbb{R}^n$ is an isomorphism. By Example 2.11, $T_{\varphi(p)}\mathbb{R}^n$ has a basis consisting of the derivations $\partial/\partial x^i|_{\varphi(p)}, i = 1, \dots, n$. Therefore the pushforwards of these vectors under $(\varphi^{-1})_*$ form a basis of $T_pM$. For ease we will use the same notation for these pushforwards, that is

$$\left.\frac{\partial}{\partial x^i}\right|_p = (\varphi^{-1})_* \left.\frac{\partial}{\partial x^i}\right|_p .$$

Thus any tangent vector $X \in T_pM$ can be written uniquely as a linear combination

$$X = \sum_{i=1}^n X^i \left.\frac{\partial}{\partial x^i}\right|_p .$$

According to the Einstein summation convention we can omit the sum sign above and write just

$$X = X^i \left.\frac{\partial}{\partial x^i}\right|_p .$$

It is easy to see, that $X^i = X(x^i)$, i.e. the effect of $X$ on the $i$-th component function, which is a smooth real-valued function on $U$.

Suppose $F\colon U \to V$ is a smooth map, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open sets. Using $(x^1, \dots, x^n)$ and $(y^1, \dots, y^m)$ to denote the coordinates on the domain and range respectively, the action of $F_*\colon T_p\mathbb{R}^m \to T_F(p)\mathbb{R}^n$ on a basis vector can be computed using the chain rule as

$$\left(F_* \left.\frac{\partial}{\partial x^i}\right|_p\right) f = \left.\frac{\partial}{\partial x^i}\right|_p (f \circ F) = \frac{\partial f}{\partial y^j}(F(p))\frac{\partial F^j}{\partial x^i}(p) = \left(\frac{\partial F}{\partial x^i}(p)\left.\frac{\partial}{\partial y^j}\right|_{F(p)}\right) f .$$

Thus

$$F_* \left.\frac{\partial}{\partial x^i}\right|_p = \frac{\partial F}{\partial x^i}(p)\left.\frac{\partial}{\partial y^j}\right|_{F(p)} .$$

This means, that $F_*$ expressed in the standard basis is

$$\begin{pmatrix} \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{pmatrix} ,$$

which is the Jacobian of $F$.

In the generally case, if $F\colon M \to N$ is a smooth map $(U, \varphi)$ and $(V, \psi)$ are coordinate charts of $p$ and $F(p)$, we obtain a coordinate representation of $\hat{F} = \psi \circ F \circ \varphi^{-1}\colon \varphi(U \cap F^{-1}(V)) \to \psi(V)$. $\hat{F}_*$ is represented with respect to the standard bases by the Jacobian matrix of $\hat{F}$. Using the fact that $F \circ \varphi^{-1} = \psi^{-1} \circ \hat{F}$ we get that

$$\begin{aligned} F_* \left.\frac{\partial}{\partial x^i}\right|_p &= F_*\left((\varphi^{-1})_* \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right) = (\varphi^{-1})_*\left(\hat{F}_* \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right) \\ &= (\psi^{-1})_*\left(\frac{\partial \hat{F}^j}{\partial x^i}(\hat{p}) \left.\frac{\partial}{\partial y^j}\right|_{\hat{F}(\varphi(p))}\right) = \frac{\partial \hat{F}^j}{\partial x^i}(\hat{p}) \left.\frac{\partial}{\partial y^j}\right|_{F(p)} , \end{aligned}$$

which means that $F_*$ is represented by the Jacobian matrix of (the coordintate representative of) $F$.

In particular, if $F = \mathrm{Id}_M$, $(U, \varphi)$ and $(V, \psi)$ are different charts of the same $p \in U \cap V$ with coordinate functions $(x^1, \dots, x^n)$ and $(\tilde{x}^1, \dots, \tilde{x}^n)$, then

$$\left.\frac{\partial}{\partial x^i}\right|_p = \frac{\partial \tilde{x}^j}{\partial x^i}(\hat{p})\left.\frac{\partial}{\partial \tilde{x}^j}\right|_p ,$$

where $\hat{p} = \varphi(p)$ is the representation of $p$ in $x^i$-coordinates. This formula is easy to remember, because it looks exactly the same as the chain rule for partial derivatives in $\mathbb{R}^n$. Applying this to the components of a vector $X = X^i \partial/\partial x^i|_p = \tilde{X}^j \partial/\partial \tilde{x}^j|_p$ we get that the transformation rule of tangent vectors is the following:

$$\tilde{X}^j = \frac{\partial \tilde{x}^j}{\partial x^i}(\hat{p}) X^i .$$

**Definition 2.16.** The *tangent bundle* of a smooth manifold $M$ is the disjoint union of the tangent spaces at all point of $M$

$$TM = \coprod_{p \in M} T_pM \ .$$

An element of $TM$ is a pair $(p, X)$, where $p \in M$ and $X \in T_pM$. Such a pair can also be denoted as $X_p$. The *natural projection* from the tangent bundle to the manifold is $\pi \colon TM \to M, (p, X) \mapsto p$.

**Definition 2.17.** A *vector field* on $M$ is a section of the map $\pi \colon TM \to M$, i.e. a map $\sigma \colon M \to TM$, such that $\pi \circ \sigma = \mathrm{Id}_M$. That is, a vector field is given by a derivation at each point of the manifold. The set of all vector fields on $M$ is denoted by $\mathcal{T}(M)$ or $\Gamma(TM)$.

## 2.2 Lie groups

**Definition 2.18.** A *Lie group* is a smooth manifold $G$ that is also a group with the property that the multiplication map $m \colon G \times G \to G$ and the inversion map $i \colon G \to G$ given by

$$m(g, h) = gh, \qquad i(g) = g^{-1},$$

are both smooth.

Because smoothness implies continuity, in particular, Lie groups are examples of topological groups.

**Example 2.19.** The following structures are Lie groups:

1. The real and complex *general linear groups* $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ consist of invertible $n \times n$ matrices with real or complex entries. These sets are groups under matrix multplication and are open submanifolds of $M(n, \mathbb{R}) \simeq \mathbb{R}^{n^2}$ and $M(n, \mathbb{C}) \simeq \mathbb{C}^{n^2}$ respectively. Multiplication and inverse are smooth maps because the elements of the results can be expressed as polynomials of the parameters.

2. The real number space $\mathbb{R}$ and Euclidean space $\mathbb{R}^n$ are Lie groups under addition, because the coordinates of $x + y$ and $-x$ are smooth (linear) function of $(x, y)$.

3. The circle $S^1 \subset \mathbb{C}^*$ is a one dimensional abelian Lie group under complex multiplication.

4. The $n$-torus $T^n = S^1 \times \cdots \times S^1$ is an $n$-dimensional abelian Lie group under complex multiplication.

**Definition 2.20.** If $G$ and $H$ are Lie groups then a smooth map $F \colon G \to H$ is a *Lie group homomorphism*, if it is also a group homomorphism. It is called *Lie group isomorphism* if it has an inverse, which is also a Lie group homomorphism.

**Example 2.21.** The map $\varepsilon \colon \mathbb{R} \to S^1$ defined by $\varepsilon(t) = e^{2\pi i t}$ is a Lie group homomorphism whose kernel is the set $\mathbb{Z}$ of integers. Similarly the map $\varepsilon^n \colon \mathbb{R}^n \to T^n$ defined by $\varepsilon^n(t_1, \ldots, t_n) = (e^{2\pi i t_1}, \ldots, e^{2\pi i t_n})$ is a Lie group homomorphism whose kernel is $\mathbb{Z}^n$.

The primary reason why Lie groups are so frequently used is that they usually appear as symmetry groups of various geometric objects.

**Definition 2.22.** A *smooth left action* of a real Lie group $G$ on a smooth manifold $M$ is a smooth map $\theta\colon G \times M \to M$, often written as $(g, p) \mapsto \theta_g(p) = g \cdot p$ such that $\theta_g\colon M \to M$ is a diffeomorphism of $M$ and for all $g_1, g_2 \in G$

$$\theta_{g_1} \circ \theta_{g_2} = \theta_{g_1 g_2} \ ,$$
$$\theta_e = \mathrm{Id}_M \ .$$

In this case $M$ is called a *smooth $G$-space*. Similarly, smooth right actions can be defined.

**Definition 2.23.** We define some concepts related to group actions. Suppose $\theta\colon G \times M \to M$ is a group action.

- For any $p \in M$, the *orbit* of $p$ under the action is the set

$$G \cdot p = \{g \cdot p : g \in G\} \ .$$

- For any $p \in M$, the *isotropy* (or *stabilizer*) *group* of $p$ is the set

$$G_p = \{g \in G : g \cdot p = p\} \ .$$

- The action is *transitive* if for any two points $p, q \in M$ there is a group element $g$ such that $g \cdot p = q$, or equivalently the orbit of any point is all of $M$.
- The action is *free* if the only element of $G$ that fixes any element of $M$ is the identity.

Let $M$ and $N$ be two $G$-spaces. A map $F\colon M \to N$ is called *equivariant* with respect to the given $G$-actions (or a *$G$-morphism*) if for each $g \in G$,

$$F(g \cdot p) = g \cdot F(p) \ .$$

Equivalently, if $\theta$ and $\varphi$ are the given actions on $M$ and $N$ respectively, $F$ is equivariant if the following diagram commutes for each $g \in G$:

$$\begin{array}{ccc} M & \xrightarrow{\ F\ } & N \\ {\scriptstyle \theta_g}\downarrow & & \downarrow{\scriptstyle \varphi_g} \\ M & \xrightarrow{\ F\ } & N. \end{array}$$

## 2.3 Lie algebras

An important concept closely related to Lie groups is that of Lie algebras. We will summarize the basic concepts here. For a more detailed treatment the reader is refered to [9, 14].

### 2.3.1 Lie bracket

Let $V$ an $W$ be smooth vector fields on $M$. In general $WV$ defined by $VWf = V(Wf)$ is not a vector field. On the other hand, it is easy to see that we can define a new smooth vector field using $V$ and $W$ by

$$[V, W]f = VWf - WVf \ ,$$

which is called the *Lie bracket* of $V$ and $W$.

**Lemma 2.24** (Properties of the Lie bracket)**.** *The Lie bracket satisfies the following identities for all $V, W, X \in \mathcal{T}(M)$:*

1. *Bilinearity: for all $a, b \in \mathbb{R}$*
$$[aV + bW, X] = a[V, X] + b[W, X] \ ,$$
$$[X, aV + bW] = a[X, V] + b[X, W] \ .$$
2. *Antisymmetry:*
$$[V, W] = -[W, V] \ .$$
3. *Jacobi identity:*
$$[V, [W, X]] + [W, [X, V]] + [X[V, W]] = 0 \ .$$

**Definition 2.25.** A *Lie algebra* is a real vector space $\mathfrak{g}$ endowed with a map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, denoted by $(X, Y) \mapsto [X, Y]$, that satisfies the properties 1–3 of Lemma 2.24.

If $G$ is a Lie group, any $g \in G$ defines a map $L_g : G \to G, h \mapsto gh$ called *left translation.* A vector field is *left-invariant* if
$$(L_g)_* X = X \ ,$$
for all $g \in G$.

**Lemma 2.26.** *If $X$ and $Y$ are smooth, left invariant vector fields on $G$, then $[X, Y]$ is also left invariant.*

**Definition 2.27.** The set of all smooth, left-invariants vector fields on $G$ endowed with the vector space operations and the Lie bracket is called the *Lie algebra of the Lie group $G$*.

The following theorem is a very deep result based on many techniques of modern differential geometry and representation theory.

**Theorem 2.28** (Lie-Cartan)**.** *There is a one-to-one correspondence between isomorphism classes of simply connected Lie groups and isomorphism classes of finite dimensional Lie algebras, which is given by associating each simply connected Lie group with its Lie algebra.*

### 2.3.2 Basic structure theory

Many of the usual algebraic concepts and methods transfer to Lie algebras.

**Definition 2.29.** A linear mapping $f : \mathfrak{g} \to \mathfrak{h}$ between two Lie algebras is a *Lie algebra homomorphism*, if for all $X, Y \in \mathfrak{g}$
$$f([X, Y]) = [f(X), f(Y)] \ .$$
If it is bijective, then $f$ is a *Lie algebra isomorphism.*

**Definition 2.30.** A subspace $\mathfrak{h} \subset \mathfrak{g}$ of a Lie algebra is a *Lie subalgebra*, if its closed under the Lie bracket, i.e. for any $X, Y \in \mathfrak{h}$ we have $[X, Y] \in \mathfrak{h}$. The subalgebra $\mathfrak{h}$ is an *ideal* if for any $X \in \mathfrak{g}, Y \in \mathfrak{h}$ we have $[X, Y] \in \mathfrak{h}$. It is easy to see that if $\mathfrak{h} \subset \mathfrak{g}$ is an ideal, then $\mathfrak{g}/\mathfrak{h}$ has a canonical structure of a Lie algebra.

**Lemma 2.31.** *If $f : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, then* Ker$f$ *is an ideal in $\mathfrak{g}$,* Im$f$ *is a subalgebra in $\mathfrak{h}$ and $f$ gives rise to an isomorphism of Lie algebras*

$$\mathfrak{g}/\mathrm{Ker}f \simeq \mathrm{Im}f \ .$$

**Definition 2.32.** We can now define some important classes of Lie algebras.

1. For a Lie algebra $\mathfrak{g}$, the *derived series* is a series of ideals defined by $D^0\mathfrak{g} = \mathfrak{g}$ and
$$D^{i+1}\mathfrak{g} = [D^i\mathfrak{g}, D^i\mathfrak{g}] \ .$$
2. The Lie algebra $\mathfrak{g}$ is *solvable* if there exist a number $n$, such that $D^n\mathfrak{g} = 0$.
3. The Lie algebra $\mathfrak{g}$ is *semisimple* if it contains no nonzero solvable ideals.
4. The Lie algebra $\mathfrak{g}$ is Abelian if $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$.
5. The Lie algebra $\mathfrak{g}$ is *simple* if it is not Abelian and contains no nontrivial ideals. If $\mathfrak{g}$ is simple then it is semisimple.

According to a theorem of Levi any Lie algebra can be written as a direct sum of semisimple Lie algebras and a unique, maximal solvable ideal called the *radical* of the Lie algebra. Therefore, it is useful to investigate the semisimple Lie algebras in more detail because they are the basic elements of which the more complex Lie algebras consist of.

**Definition 2.33.**

1. An operator $A : V \to V$ of a finite dimensional vector space is *semisimple* if any $A$-invariant subspace has an $A$-invariant complement.
2. An element $X \in \mathfrak{g}$ is *semisimple* if the linear operator
$$\mathfrak{g} \to \mathfrak{g}, Y \mapsto [X, Y]$$
is semisimple.
3. A Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is *toral*, if it Abelian and consists of semisimple elements.
4. Let $\mathfrak{g}$ be a semisimple Lie algebra. A *Cartan subalgebra* is toral subalgebra $\mathfrak{h} \subset \mathfrak{g}$ which coincides with its centralizer:
$$Z(\mathfrak{h}) = \{X : [X, \mathfrak{h}]\} = h \ .$$

It can be shown, that the maximal toral subalgebras are always Cartan subalgebras.

From now on we fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

**Theorem 2.34.** *Every Lie algebra $\mathfrak{g}$ has a decomposition,*

$$\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \ ,$$

*where*

$$g_\alpha = \{X : [H, X] = \alpha(H)X, \quad \forall H \in \mathfrak{h}\} \ ,$$
$$R = \{\alpha \in \mathfrak{h}^* \setminus \{0\} : \mathfrak{g}_\alpha \neq 0\} \ .$$

*This is called the root decomposition of $\mathfrak{g}$.*

### 2.3.3 Root systems and Dynkin diagrams

**Definition 2.35.** Let $V$ be a finite dimensional Euclidean space with inner product $\langle\cdot,\cdot\rangle \to \mathbb{R}$. An *abstract root system* is a finite set of elements $R \subset V \setminus \{0\}$, such that

(a) $R$ generates $V$.

(b) For any two roots $\alpha, \beta$ the number

$$n_{\alpha\beta} = \frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}$$

is an integer.

(c) Let $s_\alpha : V \to V$ be defined by

$$s_\alpha(\lambda) = \lambda - \frac{2\langle\alpha,\lambda\rangle}{\langle\alpha,\alpha\rangle}\alpha\ .$$

Then $s_\alpha(\beta) \in R$ for all $\alpha, \beta \in R$.

$R$ is a *reduced* root system, if it satisfies the additional property that when $\alpha, c\alpha$ are both roots then $c = \pm 1$.

An important result is that for each semisimple complex Lie algebra a root system can be associated.

**Theorem 2.36.** *Let $\mathfrak{g}$ be a semisimple complex Lie algebra, with root decomposition given by Theorem 2.34. The the set of roots $R \subset \mathfrak{h}^* \setminus \{0\}$ is a reduced root system with an appropriate inner product on $\mathfrak{h}^*$.*

It is an important theorem of Chevalley and Serre that a reduced root system $R$ defines a complex Lie algebra with the so-called Serre relations. This Lie algebra is semisimple and has root system naturally isomorphic to $R$. Therefore, there is a natural bijection between the set of isomorphism classes of reduced root systems and the set of isomorphism classes of (finite dimensional) complex semisimple Lie algebra. Due to this correspondence, the classification of semisimple Lie algebras is reduced to the easier task of classifying reduced root systems.

The definition of abstract root systems implies very strong restrictions on the relative position of two roots.

**Theorem 2.37.** *Let $\alpha, \beta \in R$ be two roots which are not multiples of each other with $|\alpha| \geq |\beta|$ and let $\varphi$ be the angle between them. Then only one of the following configurations are possible:*

*(a)* $\varphi = \pi/2$,

*(b)* $\varphi = 2\pi/3, |\alpha| = |\beta|$,

*(c)* $\varphi = \pi/3, |\alpha| = |\beta|$,

*(d)* $\varphi = 3\pi/4, |\alpha| = \sqrt{2}|\beta|$,

*(e)* $\varphi = \pi/4, |\alpha| = \sqrt{2}|\beta|$,

*(f)* $\varphi = 5\pi/6, |\alpha| = \sqrt{3}|\beta|$,

*(g)* $\varphi = \pi/6, |\alpha| = \sqrt{3}|\beta|$.

It can be shown, that each of these possibilities is indeed realized.

**Definition 2.38.** Choose a $t \in V$ such that $\langle t, \alpha \rangle \neq 0$ for any root $\alpha$. Then

$$R = R_+ \sqcup R_- ,$$

where

$$R_+ = \{\alpha \in R : \langle \alpha, t \rangle > 0\}, \quad R_- = \{\alpha \in R : \langle \alpha, t \rangle < 0\} .$$

This is called a *polarization* of $R$. Members of the set $R_+$ are called the *positive roots.* A positive root $\alpha \in R_+$ is *simple* if it cannot be written as a sum of two positive roots.

For reasons we cannot go into here, for each root system $R$ and polarization $R = R_+ \sqcup R_-$ the root system can be recovered form the set simple roots $\Pi = \{\alpha_1, \ldots, \alpha_r\}$. Moreover, different sets of simple roots given by different choices of polarizations are related to each other with a special symmetry group, called the *Weyl group.* Thus, classifying root systems is equivalent to classifying possible sets of simple roots [14].

**Definition 2.39.** A root system $R$ is *reducible* if it can be written in the form $R = R_1 \sqcup R_2$, where $R_1 \perp R_2$. Otherwise, $R$ is *irreducible.*

**Definition 2.40.** Let $\Pi$ be a set of simple root sof a root system $R$. The *Dynkin diagram* of $\Pi$ is the graph constructed according to the following steps:

1. For each simple root $\alpha_i$ a vertex $v_i$ is associated.

2. For each pair of simple roots $\alpha_i \neq \alpha_j$, with $\varphi = \langle \alpha_i, \alpha_j \rangle$ connect the corresponding vertices with $n$ edges according to the following rules

   (a) If $\phi = \pi/2$, then $n = 0$.

   (b) If $\phi = 2\pi/3$, then $n = 1$.

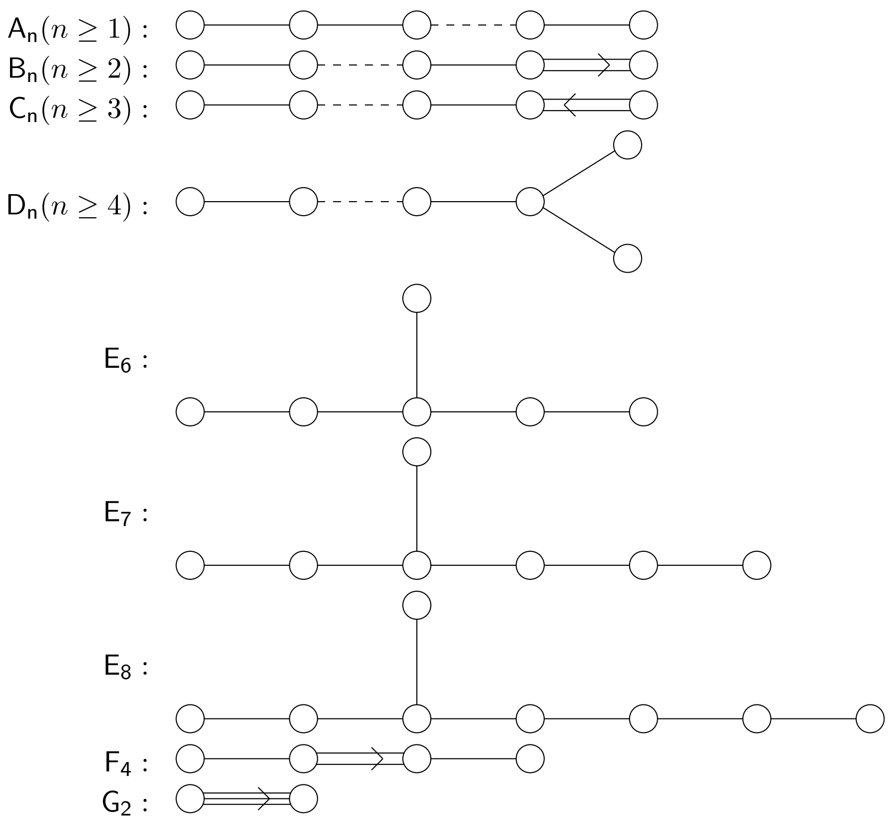
   (c) If $\phi = 3\pi/4$, then $n = 2$.

   (d) If $\phi = 5\pi/6$, then $n = 3$.

   Because we consider only the simple roots, these are the only possible configurations of Theorem 2.37.

3. For every pair of distinct simple roots $\alpha_i \neq \alpha_j$, if $|\alpha_1| \neq |\alpha_j|$ and they are not orthogonal, we orient the corresponding edges to point towards the shorter root.

**Theorem 2.41.** *If $R$ is a reduced irreducible root system, then its Dynkin diagram is one of the followings ($n$ vertices in each case):*




With some algebraic tricks the following statement can also be proved.

**Proposition 2.42.** *The Lie algebra is simple if and only if its root system is irreducible.*

**Corollary 2.43.** *Combining this with the mentioned bijection between the isomorphism classes of complex semisimple Lie algebras and reduced root systems we conclude that simple finite-dimensional complex Lie algebras are classified by Dynkin diagrams $A_n, \ldots, G_2$ listed in Theorem 2.41.*

It is common to refer to the simple Lie algebra corresponding to the Dynkin diagram e.g. $\mathsf{G}_2$ as the simple Lie algebra of type $\mathsf{G}_2$, or the simple Lie algebra $\mathfrak{g}_2$. The Lie algebras $\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{f}_4$ and $\mathfrak{g}_2$ are called *exceptional*, because contrary to the Lie algebras of other types they are not members of infinite sequences.

## 2.4 Principal bundles

Suppose $G$ is a compact Lie group.

**Definition 2.44.** A locally trivial fibre bundle $\pi\colon P \to M$ is called a *$G$-bundle* if the followings are satisfied:

1. $\pi$ is a smooth map between manifolds.

2. $P$ is a $G$-space.

3. The action of $G$ preserves the fibers, i.e. $y \in P_x$ implies $g \cdot y \in P_x$ for all $g \in G$.

This means that in such cases the fibres themselves are $G$-spaces as well. If $\pi\colon P \to M$ is a $G$-bundle, then for each $g \in G$ the action of $g$ is a bundle automorphism. That is, the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\theta_g} & P \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & M. & \end{array}$$

The most importat kind of $G$-bundles are principal bundles.

**Definition 2.45.** A $G$-bundle $\pi\colon P \to M$ is a *principal $G$-bundle*, if the action of $G$ on the fibers is free and transitive [3, 10, 15]. This implies that the orbits of the action are the fibers of the bundle, and these fibers as spaces are diffeomorphic to $G$ (but generally the fibers are not groups). Moreover, the orbit space $P/G$ is diffeomorphic to $M$.

*Remark.* The notion of $G$-spaces and principal bundles can be defined also in the category of topological manifolds. In this case the group $G$ is an arbitrary topological group and the fibers are homeomorphic to $G$.

It follows from the definition, that similarly to vector bundles if $(U_\alpha, \Phi_\alpha)$ and $(U_\beta, \Phi_\beta)$ are two overlapping local trivializations of $P$ then one has

$$\Phi_{\alpha\beta}(p, g) = \Phi_\beta \circ \Phi_\alpha^{-1}(p, g) = (p, \tau_{\alpha\beta}(p)g) \ .$$

This means that just like vector bundles, the principal $G$-bundles have transition function $\tau_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ between local trivializations. In particular, the functions $\tau_{\alpha\beta}$ for a principal bundle satisfy the same cocycle conditions (2.1) as the transition functions of vector bundles.

A principal $G$-bundle over $M$ can be obtained from a family of functions $\tau_{\alpha\beta}$, corresponding to an open cover of $M$ and satisfying (2.1) in the following way. For each trivializing neighbourhood $U_\alpha \subset M$ consider $U_\alpha \times G$. Define an equivalence relation between element $(p, h) \in U_\alpha \times G$, $(p', h') \in U_\beta \times G$, so that $(p, h) \sim (p', h')$ if and only if $p' = p$ and $h' = \tau_{\alpha\beta}(p)h$. Now let

$$P = \coprod_\alpha (U_\alpha \times G)/\sim \ ,$$

which is the disjoint union of the product sets $U_\alpha \times G$ glued together according to the equivalence relation. It can be shown that the resulting $P$ together with the projection mapping is a principal $G$-bundle.

*Remark.* If the group $G$ is a matrix Lie group, then the transition functions can be taken from some vector bundle $E$ over $M$. In this way $P$ is constructed from $E$. The construction can be done in the other direction as well. That is, one starts with a principal $G$-bundle over a base manifold $M$ and obtains a vector bundle $E$ over $M$. This vector bundle $E$ is the *associated bundle* of $P$. In either case tha data of transition functions is the same for the principal $G$-bundle $P$ and the vector bundle $P$. The difference in the action of the subgroup $G$ on the fibres. In the principal $G$-bundle case $G$ acts on itself by left translation, while in the vector bundle case it acts as a subgroup of $GL(k, \mathbb{R})$ on $\mathbb{R}^k$.

### 2.4.1 Milnor's construction and homotopic classification

Let $\pi\colon P \to N$ be a principal bundle, and $f\colon M \to N$ a smooth map. The pullback of $P$ is a bundle $f^*P \to M$ is

$$f^*P = \{(p,x) \in M \times P : f(p) = \pi(x)\} ,$$

i.e. we put the fiber $P_{f(p)} = \pi^{-1}(f(p))$ on $p \in M$. This is clearly a principal bundle if we define the left action by $g \cdot (p,x) = (p, g \cdot x)$. The projection onto the second component gives a $G$-equivariant map $F : f^*P \to P$ such that the following diagram commutes

$$\begin{array}{ccc} f^*P & \xrightarrow{\;F\;} & P \\ \pi \big\downarrow & & \big\downarrow \pi \\ M & \xrightarrow{\;f\;} & N. \end{array}$$

**Theorem 2.46.** *Let $\pi\colon P \to N$ be a principal $G$-bundle and $f_t\colon M \to N$ be a homotopy. Then the principal $G$-bundles $f_0^*P$ and $f_1^*P$ are isomorphic over $M$.*

**Definition 2.47.** A *universal bundle* for a group $G$ is a principal $G$-bundle $E_G \to B_G$, such that for every principal $G$-bundle $\pi\colon P \to M$ there exists a smooth map $f : M \to B_G$ $P \simeq f^*E_G$.

It can be shown that the *classifying space* $B_G$ and the universal bundle $E_G$ is unique up to homotopy type. The construction of an explicit universal bundle was given by Milnor.

**Definition 2.48.** The *join* of the spaces $X$ and $Y$ is defined as $X * Y = X \times Y \times I/\sim$, where $(x,y,0) \sim (x,y',0)$ for all $y,y' \in Y$ and $(x,y,1) \sim (x',y,1)$ for all $x,x' \in X$. That is, all points in $X$ are connected with all points in $Y$ with a unit length line.

$E_G$ is defined as a limit of a sequence of spaces. The $n$-th space of this sequence is

$$E_G(n) = \overbrace{G * \cdots * G}^{n} = \left\{g_1, \ldots, g_n, t_1, \ldots, t_n : \sum t_i = 1, t_i \geq 0, g_i \in G\right\} .$$

Then, $E_G(n-1) \subset E_G(n)$ with $t_n = 0$ and thus we take the direct limit

$$E_G = \lim_{n\to\infty} E_G(n) .$$

The group $G$ acts on $E_G$ in a natural way: the effect of $g$ is multiplication with itself on each $g_i$. This brings fibers to fibres, the effect on the fibers is free and transitive. Therefore, the factor map

$$E_G \to B_G = E_G/G$$

makes $E_G$ a principal bundle over $B_G$.

The proof of the following statements can be found in [10].

**Theorem 2.49.** *If $M$ is a paracompact manifold and $P \to M$ is a principal $G$-bundle, then there exists a map $f : M \to B_G$ such that $P \simeq f^*E_G$, and this $f$ is unique up to homotopy.*

**Theorem 2.50.** *If $f_1, f_2 : M \to B_G$ two maps such that $f_1^*E_G \simeq f_2^*E_G$, then $f_0$ and $f_1$ are homotopic.*

**Corollary 2.51.** *Together with Theorem 2.46 these imply that there is a one-to-one correspondence between the principal $G$-bundles over $B$ and $[M, B_G]$, the homotopy classes of the maps from $M$ to $B_G$. If $M$ is a sphere $S^n$, then $[S^n, B_G] = \pi_n(B_G)$. Due to the construction the space $E_G$ is contractible, and therefore $\pi_n(E_G) = 0$ for all $n > 0$. As a consequence, using the homotopy equact sequence of the fibration $E_G \xrightarrow{G} B_G$ this means that*

$$\underbrace{\pi_n(E_G)}_{0} \longrightarrow \pi_n(B_G) \xrightarrow{\simeq} \pi_{n-1}(G) \longrightarrow \underbrace{\pi_{n-1}(E_G)}_{0},$$

*i.e. there is a one-to-one correspondence between the locally trivial principal $G$-bundles over $S^n$ and $\pi_{n-1}(G)$.*

# Chapter 3

# Division algebras over the reals

This chapter starts by introducing the concept of doubling of algebras, which leads to higher dimensionsal structures analogous with the real and complex numbers. After deriving some general facts, the next step in this construction, namely the quaternions are used to derive the transition functions of a toy principal bundle. Finally, we take a further step to get the octonion algebra.

## 3.1 The Cayley-Dickson construction

Let $\mathcal{A}$ be a not necessarily associative (but finite-dimensional) algebra over the field $\mathbb{R}$. A linear mapping $a \mapsto \bar{a}$ of $\mathcal{A}$ to itself is said to be a *conjugation* or *involutory anti-automorphism* if $\bar{\bar{a}} = a$ and $\overline{ab} = \bar{b}\bar{a}$ for any elements $a, b \in A$ (the case $\bar{a} = a$ is not excluded).

**Definition 3.1** (Cayley-Dickson construction [19], [1])**.** Consider the vector space of the direct sum of two copies of an algebra: $\mathcal{A}^2 = \mathcal{A} \oplus \mathcal{A}$. A multiplication on $\mathcal{A}^2$ is defined as:

$$(a, b)(u, v) = (au - \bar{v}b, b\bar{u} + va) \ .$$

It is easy to check, that relative to this multiplication the vector space $\mathcal{A}^2$ is an algebra of dimension $2 \cdot \dim(\mathcal{A})$. This is called the *doubling* of the algebra $\mathcal{A}$.

*Remark.* The correspondence $a \mapsto (a, 0)$ is a monomorphism of $\mathcal{A}$ into $\mathcal{A}^2$. Therefore we will identify elements $a$ and $(a, 0)$ and thus assume $\mathcal{A}$ is a subalgebra of $\mathcal{A}^2$. If $\mathcal{A}$ has an identity element, then the element $1 = (1, 0)$ is obviously an identity element in $\mathcal{A}^2$.

An important element in $\mathcal{A}^2$ is $e = (0, 1)$. It follows from the definition of multiplication that $be = (0, b)$ and hence $(a, b) = a + be$ for all $a, b \in \mathcal{A}$. Thus every element of the algebra $\mathcal{A}^2$ is uniquely written as $a + be$. Moreover, as it can be easily checked the following indentities are true:

$$a(be) = (ba)e, \qquad (ae)b = (a\bar{b})e, \qquad (ae)(be) = -\bar{b}a \ . \tag{3.1}$$

In particular $e^2 = -1$.

To iterate the Cayley-Dickson construction it is necessary to define a conjugation in $\mathcal{A}^2$. This will be done by the formula

$$\overline{a + be} = \bar{a} - be \ .$$

This is involutory, $\mathbb{R}$-linear and is simultaneously an antiautomorphism. Using this definition, the doubling $\mathbb{R}^2$ of the field $\mathbb{R}$ is the algebra $\mathbb{C}$ of complex numbers and the doubling

$\mathbb{C}^2$ of $\mathbb{C}$ is the algebra of quaternions $\mathbb{H}$. In the latter case $e$ is denoted by $j$ and $ie$ is denoted by $k$, and thus a general quaternion is of the form $r = r_1 + r_2 i + r_3 j + r_4 k$, where $r_i \in \mathbb{R}, i = 1, 2, 3, 4$. Due to the identities (3.1) $ea = \bar{a}e$ for all $a \in \mathcal{A}$. Therefore, $\mathcal{A}^2$ is not commutative if the original conjugation is not the identity mapping. In particular $\mathbb{H}$ is not commutative.

### 3.1.1 Some classes of algebras

**Definition 3.2.** An algebra with an identity element over the field $\mathbb{R}$ is a *metric algebra*, if $a\bar{a} \in \mathbb{R} = 1 \cdot \mathbb{R}$, and $a\bar{a} > 0$ for $a \neq 0$. In this case the real number $|a| = \sqrt{a\bar{a}}$ is called the *norm of a*. By definition $|a| = 0$ if and only if $a = 0$.

**Lemma 3.3.** *If $\mathcal{A}$ is a metric algebra then $\mathcal{A}^2$ is also metric.*

*Proof.* For any $a, b \in \mathcal{A}$

$$(a + be)\overline{(a + be)} = (a + be)(\bar{a} - be) = a\bar{a} + be\bar{a} - abe + b\bar{b} = a\bar{a} + b\bar{b} ,$$

since $be\bar{a} = abe$ according to the rules of multiplication in (3.1). Therefore, $(a + be)\overline{(a + be)} \in \mathbb{R}$ and it is obvously positive if $a$ or $b$ is not 0. □

In any metric algebra the formula

$$\langle x, y\rangle = \frac{x\bar{y} + y\bar{x}}{2}$$

defines a scalar product, and the algebra is an Euclidean space with respect to this scalar product. The orthogonal complent of the identity element in a metric algebra is denoted by $\mathcal{A}'$. Consequently, all $a \in \mathcal{A}$ can be uniquely written as $a = \lambda + a'$, where $\lambda \in \mathbb{R}$ and $a' \in \mathcal{A}'$, and therefore $\bar{a} = \lambda - a'$. Thus, $a \in \mathcal{A}'$ if and only if $\bar{a} = -a$, and $a \in \mathbb{R}$ if and only if $\bar{a} = a$. Since by the definition $x\bar{y} + \bar{x}y = 2\langle x, y\rangle$ for any elements $x, y \in \mathcal{A}$, $yx = -xy$ if and only if $x \perp y$.

*Remark.* (a) Using again the rules of multiplication the scalar product in $\mathcal{A}^2$ is

$$\langle a + be, u + ve\rangle = \langle a, u\rangle + \langle b, v\rangle ,$$

and this means that $\mathcal{A}^2 = \mathcal{A} \oplus \mathcal{A}$ as an orthogonal direct sum of vector spaces.

(b) In any metric algebra for an orthogonal operator $\Phi(\bar{a}) = \overline{\Phi(a)}$, because $\Phi(1) = 1$, and thus $\Phi(\mathcal{A}') = \mathcal{A}'$.

**Definition 3.4.** The algebra $\mathcal{A}$ is a *normed algebra* (or *composition algebra*), if there is a norm $|.|$ over the vector space $\mathcal{A}$ such that

$$|ab| = |a||b| .$$

**Definition 3.5.** The algebra $\mathcal{A}$ is a *division algebra* if the equations $ax = b$ and $xa = b$ are uniquely solvable for all nonzero $a, b \in \mathcal{A}$.

**Lemma 3.6.** *If $\mathcal{A}$ is a normed algebra then it is a division algebra.*

*Proof.* Since $|ab| = |a||b|$, the mappings $x \mapsto \frac{ax}{|x|}$ and $x \mapsto \frac{xa}{|x|}$ are isometries for all $a \neq 0$, hence they are bijective ($\mathcal{A}$ is finite dimensional). The solutions of the equations $ax = b$ and $xa = b$ are given by $x = a^{-1}b$ and $x = ba^{-1}$ respectively, where $a^{-1} = \frac{\bar{a}}{|a|^2}$. □

Let $\mathcal{A}$ be a finite dimensional algebra over the field $\mathbb{R}$ with basis $e_1, \ldots, e_n$. An invertible linear mapping $\Phi : \mathcal{A} \to \mathcal{A}$ is an automorphism if and only if $\Phi(e_i e_j) = \Phi(e_i)\Phi(e_j)$ for any $i, j = 1, \ldots, n$. Consequently if $\Phi(e_i) = x_i^j e_j$ and $e_i e_j = c_{ij}^k e_k$ and hence

$$\Phi(e_i e_j) = \Phi(c_{ij}^k e_k) = c_{ij}^k x_k^l e_l \ ,$$

$$\Phi(e_i)\Phi(e_j) = (x_i^p e_p)(x_j^q e_q) = c_{pq}^l x_i^p x_j^q e_l \ ,$$

then $\Phi$ is an automorphism if and only if

$$c_{ij}^k x_k^l = c_{pq}^l x_i^p x_j^q$$

for any $i, j = 1, \ldots, n$. This condition gives a set of polynomial equations for the elements of the matrices $(x_i^j)$ corresponding to the automorphisms of $\mathcal{A}$. Therefore, the set of all automorphisms is an algebraic variety in the space $\mathbb{R}^{n^2}$ which is obviously closed.

**Theorem 3.7** (Cartan). *If $G$ is a Lie group, and $H \subset G$ is a closed subgroup, then there exists a unique smooth structure on $H$ which makes it an embedded Lie subgroup of $G$, i.e. a Lie subgroup, which is an embedded submanifold.*

**Corollary 3.8.** *Giving a basis $e_1, \ldots, e_n$ in $\mathcal{A}$ defines an isomorphism between the group of all automorphism of an algebra, denoted by* $\operatorname{Aut}\mathcal{A}$ *and a matrix Lie group. It can be shown that the Lie group structure induced on* $\operatorname{Aut}\mathcal{A}$ *is independent of the choice of the basis. Therefore,* $\operatorname{Aut}\mathcal{A}$ *is a Lie group.*

Similarly to the derivations of a manifold it is possible to define derivations of an algebra.

**Definition 3.9.** A *derivation* of an algebra $\mathcal{A}$ is an $\mathbb{R}$-linear map $D : \mathcal{A} \to \mathcal{A}$, such that

$$D(ab) = aD(b) + D(a)b \ .$$

The set of all derivation of $\mathcal{A}$ is denoted by $\operatorname{Der}\mathcal{A}$. It is clearly a vector space over $\mathbb{R}$.

The next proposition shows, that there is a nice characterization of the Lie algebra of $\operatorname{Aut}\mathcal{A}$ using the derivations.

**Proposition 3.10.** *The Lie algebra of the Lie group* $\operatorname{Aut}\mathcal{A}$ *is the vector space* $\operatorname{Der}\mathcal{A}$*, endowed with the commutator as the Lie bracket:*

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1 \ .$$

According to Corollary 3.8, $\operatorname{Aut}\mathcal{A} \subset GL(n, \mathbb{R})$ where $n = \dim \mathcal{A}$. In case $\mathcal{A}$ is normed this statement can be made stronger.

**Lemma 3.11.** *Let $\mathcal{A}$ be a normed algebra and let $\Phi : \mathcal{A} \to \mathcal{A}$ be an automorphism of it. Then $\Phi$ is an orthogonal operator, i.e. $|\Phi(a)| = |a|$.*

*Proof.* For all $a \in \mathcal{A}$,

$$\Phi(a) = \Phi\left(|a| \cdot \frac{a}{|a|}\right) = \Phi(|a|)\Phi\left(\frac{a}{|a|}\right) = |a|\Phi\left(\frac{a}{|a|}\right) \ .$$

Therefore, it is enough to proove that if $|a| = 1$, then $|\Phi(a)| = 1$. Suppose $|\Phi(a)| < 1$. Then, $|\Phi(a^k)| = |(\Phi(a))^k| = |\Phi(a)|^k \to 0$ as $k \to \infty$. Thus, $|a|^k \to 0$ as $k \to \infty$, which is impossible for $|a| = 1$. Similarly, if $|\Phi(a)| > 1$, then $|a|^k \to \infty$ as $k \to \infty$, which is also impossible. Hence $|\Phi(a)| = 1$. □

**Corollary 3.12.** *In any basis of $\mathcal{A}$, the matrix of an automorphism $\Phi$ is representented by an element of the group $O(n)$ of orthogonal matrices, where $n = \dim \mathcal{A}$. Since $\Phi(1) = 1$ always, $\Phi(\lambda + a') = \lambda + \Phi'(a')$ for some $\Phi' : \mathcal{A} \to \mathcal{A}'$. The operators $\Phi$ and $\Phi'$ can be identified and thus*

$$\operatorname{Aut} \mathcal{A} \subset O(n-1) \ .$$

## 3.2 Quaternions

As we mentioned above, the doubling of $\mathbb{C}$ is the quaternion algebra $\mathbb{H}$ which is associative but not commutative. It is a 4 dimensional algebra with basis $\{1, i, j, k\}$. The multiplication rules are encapsulated by the relations

$$i^2 = j^2 = k^2 = -1 \ ,$$

$$ij = k, \qquad ji = -k, \qquad jk = i, \qquad kj = -i, \qquad ki = j, \qquad ik = -j \ .$$

### 3.2.1 The Hopf map

A quaternion $r$ determines a linear mapping $R_r : \mathbb{R}^3 \to \mathbb{R}^3$ in the following way. To a point $p = (x, y, z)$ in the 3-space, we associate a quaternion $xi + yj + zk$. In this way $\mathbb{R}^3$ is identified with the set of *imaginary* (or *pure*) quaternions, i.e. the quaternions that have real part zero which is $\mathbb{H}'$. It is easy to see that for any $r \in \mathbb{H}$, $p \in \mathbb{H}'$ the quaternion product $rpr^{-1}$ is also pure and the mapping

$$R_r : \mathbb{H}' \to \mathbb{H}', \quad p \to rpr^{-1}$$

is linear over $\mathbb{R}$. Moreover, with some calculation it can be shown that it is a rotation around the axis $(r_2, r_3, r_4)$ with angle $\theta = 2\cos^{-1}(r_1)$. Because of associativity any $r \in \mathbb{H}$ generates an automorphism of the quaternions by the mapping $x \mapsto rxr^{-1}$. These automorphisms are called *inner automorphisms*. If $|r| \neq 1$, then by exchanging $r$ to $r/|r|$ one gets the same rotation. Therefore, all the rotations of the 3-dimensional Euclidean space can be desribed with the set of unit length quaternions

$$\{r_1 + r_2 i + r_3 j + r_4 k : r_1^2 + r_2^2 + r_3^2 + r_4^2 = 1\}$$

which coincide with $S^3$, the unit sphere in $\mathbb{R}^4$.

**Proposition 3.13.** *The group of automorphisms of the quaternion algebra is* $\operatorname{Aut} \mathbb{H} = SO(3)$*.*

*Proof.* According to Corollary 3.12 $\operatorname{Aut} \mathbb{H} \subset O(3)$. Since an orientation reversing mapping cannot preserve the equation $ij = k$, we have that $\operatorname{Aut} \mathbb{H} \subset SO(3)$. But all the elements of $SO(3)$ occur as inner automorphisms. □

Geometrically this can be seen by observing that $i, j$ and $k$ can be mapped to any positively oriented orthonormal basis of $\mathbb{H}'$. A detailed proof is in [19].

The Hopf map can be formulated with the help of the quaternions. First, fix a distinguished point, e.g. $N = (1, 0, 0) \in S^2$. A unit length quaternion $r \in S^3$ defines a rotation. Then the Hopf fibration is defined by

$$S^3 \to S^2, \qquad r \mapsto R_r(N) = ri\bar{r} \ .$$

Now consider the points which map to $N \in S^2$. It is easy to check, that the set of points

$$\{(\cos t \sin t, 0, 0) : t \in [0, 2\pi]\}$$

in $S^3$ all map to $(1, 0, 0)$ via the Hopf map. In fact, this set is the entire set of points that map to $(1, 0, 0)$, and therefore it is the preimage of $(1, 0, 0)$. Similarly, the preimage of an arbitrary $v = (v_1, v_2, v_3) \in S^2 \setminus \{S\}$ is

$$\left\{ \sqrt{\frac{1+v_1}{2}} \left( \cos t, \sin t, \frac{v_2 \sin t - v_3 \cos t}{1+v_1}, \frac{v_2 \cos t + v_3 \sin t}{1+v_1} \right) : t \in [0, 2\pi] \right\} .$$

### 3.2.2 $SO(3)$ as an $SO(2)$-bundle over $S^2$

Consider the Lie group $SO(3) = \operatorname{Aut} \mathbb{H}$. This group consists of $3 \times 3$ orthogonal real matrices with determinant 1. In other words, the elements of this group are the rotations of a 3-dimensional Euclidean space. If $A \in SO(3)$, then $A^T A = AA^T = I$ and the columns of $A$ are the images of $i, j$ and $k$, the standard bases. Consider the mapping

$$p \colon SO(3) \to S^2, \quad [v_1|v_2|v_3] \mapsto v_1 \ ,$$

where $v_1, v_2$ and $v_3$ are column vectors of a $3 \times 3$ matrix. That is, we evaluate an automorphism $\varphi : \mathbb{H} \to \mathbb{H}$ at $i$:

$$p \colon \operatorname{Aut} \mathbb{H} \to S^2, \quad \varphi \mapsto \varphi(i) \ .$$

The preimage of a given unit length vector $v_1 \in S^2$ consists of triples of unit length vectors which are columns of the orthogonal matrices having $v_1$ as their first columns:

$$p^{-1}(v_1) = \{[v_1|v_2|v_3], v_2 \perp v_1, v_3 = v_1 \times v_2, |v_2|^2 = |v_3|^2 = 1\}$$

Since the vector $v_3$ is determined by $v_1$ and $v_2$, we need to describe only the latter two. The vector $v_2$ should lie in the normal plane of $v_1$ and since it has length 1, it can take its values from $S^1 \subset V_{v_1}$, where $V_{v_1}$ is the tangent space to $v_1$. Taking into acount the orientation $v_2$ and $v_3$ determine an element of $SO(2)$. The effect of $SO(3)$ on $S^2$ is clearly transitive, i.e. $i$ can be taken to any vector in $S^2$. The preimage of $i$ consis of matrices that leave $i$ fixed. That is, $p^{-1}(i)$ is isomorphic to $SO(2)$ by deleting the first column and the first row of the matrices. $SO(3)$ acts on itself by left multiplicaiton and using this identification $SO(2)$ acts on $SO(3)$. Because it is transitive and free on $p^{-1}(i)$, the same is true for all of the fibers. Due to the standard theorem of transitive Lie group actions ([17], Theorem 9.24) the next statement follows.

**Proposition 3.14.** *$SO(3)$ is a principal $SO(2)$-bundle over $S^2$, and*

$$SO(3)/SO(2) \approx S^2 \ .$$

In particular, the fibration $p \colon SO(3) \to S^2$ is locally trivial. We will calculate explicitly the trivializing functions since this will be useful in the case of the group $G_2$, which will be presented in the next Chapter. We will use two charts on the base space, namely $S^2 \setminus \{S\}$ and $S^2 \setminus \{N\}$ (these are homeomorphic to $\mathbb{R}^2$ using the stereographic projection). As mentioned above, if $\varphi \in p^{-1}(i)$, i.e. $\varphi(i) = v_1 = i$, then $\varphi(j) = v_2$ and $\varphi(k) = v_3$ lie in the plane $jk$, and the correspondance between $p^{-1}(i)$ and $SO(2)$ is straighforward: because $\varphi$ restricts to a linear operator on $V_i$ we assign to it the matrix of this mapping in the basis

$\{j, k\}$. That is, we just forget the 0 at the first coordinate of $v_2$ and $v_3$. Let us call this $p^{-1}(i) \to SO(2)$ diffeomorphism $\theta_i$.

For a general $v \in S^2 \setminus \{S\}$ any $\varphi \in p^{-1}(v)$ restricts to a $V_i \to V_v$ mapping. We will choose a basis in $V_v$ and write the images of $j$ and $k$ in this basis. To find a basis in $V_v$ we will define a translating automorphism $Q_v$ such that $Q_v(i) = v$ and therefore $Q_v(j), Q_v(k) \in T_v$. Define $\theta_v \colon p^{-1}(v) \to SO(2)$ by assigning to each automorphism $\varphi \in p^{-1}(v)$ the matrix of its restriction $V_i \to V_v$ written in the bases $\{j, k\}$ at $V_i$ and $\{Q_v(j), Q_v(k)\}$ at $V_v$. Using this a trivializing map is given by

$$\psi_1 \colon p^{-1}(S^2 \setminus \{S\}) \to S^2 \setminus \{S\} \times SO(2), \quad \varphi \mapsto (\varphi(i), \theta_{\varphi(i)}(\varphi)) .$$

Similarly, if $\varphi \in p^{-1}(-i)$ then $\varphi(j)$ and $\varphi(k)$ are in the plane $jk$. The triple $(-i, \varphi(j), \varphi(k))$ form an orthonormal basis of $\mathbb{R}^3$ which has the same orientation as the standard basis. This means that restricted to the plane $jk$ the vectors $(\varphi(j), \varphi(k))$ change the standard orientation given by $(j, k)$. Thus, the parametrization of $p^{-1}(-i)$ is the following: we just forget the 0 at the first coordinate of $v_2$ and $-v_3$. That is, we assing to $\varphi$ the matrix of the mapping $V_{-i} \to V_{-i}$ induced by $\varphi$ written in the basis $\{-i, j, -k = -ij\}$. This $p^{-1}(-i) \to SO(2)$ diffeomorphism will be called $\tilde{\theta}_{-i}$. As we did in the previous case, for a general $v \in S^2 \setminus \{N\}$ we will choose a translating automorphism $\tilde{Q}_v$ such that $\tilde{Q}_v(-i) = v$ and therefore $\tilde{Q}_v(j), \tilde{Q}_v(-k) \in V_v$. Then we define $\tilde{\theta}_v \colon p^{-1}(v) \to SO(2)$ by assigning to $\varphi \in p^{-1}(v)$ the matrix of the corresponding linear mapping from $V_{-i}$ onto $V_v$ written in the bases $\{j, -k\}$ at $V_{-i}$ and $\{\tilde{Q}_v(j), \tilde{Q}_v(-k)\}$ at $V_v$. Again, a trivializing map is given by

$$\psi_2 \colon p^{-1}(S^2 \setminus \{N\}) \to S^2 \setminus \{N\} \times SO(2), \quad \varphi \mapsto (\varphi(i), \tilde{\theta}_{\varphi(i)}(\varphi)) .$$

Next, we need to find translating automorphisms $Q_v$ and $\tilde{Q}_v$. We require $\tilde{Q}_v(-i) = v$ which is equivalent to $-\tilde{Q}_v(-i) = -v = Q_{-v}(i) = -Q_{-v}(-i)$. Therefore it is enough find $Q_v$, because then $\tilde{Q}_v = Q_{-v}$ satisfies the required condition for the second translating automorphism. We will look for $Q_v$ in the form of an inner automorphism generated by a unit length quaternion $r_v \in \mathbb{H}$. For a quaternion $r = r_1 + r_2 i + r_3 j + r_4 k = (r_1, r_2, r_3, r_4)$ we have the Hopf map:

$$rir^{-1} = ri\bar{r} = (0, r_1^2 + r_2^2 - r_3^2 - r_4^2, 2(r_2 r_3 + r_1 r_4), 2(r_2 r_4 - r_1 r_3)) .$$

Since we need $r_v i \bar{r}_v = v$, we shall solve the following system of equations:

$$r_1^2 + r_2^2 - r_3^2 - r_4^2 = v_1 \tag{3.2}$$
$$2(r_2 r_3 + r_1 r_4) = v_2 \tag{3.3}$$
$$2(r_2 r_4 - r_1 r_3) = v_3 \tag{3.4}$$

The solution of this system for a vector $v \in S^2 \setminus \{S\}$ is

$$r_v = \sqrt{\frac{1 + v_1}{2}} \left( \cos t, \sin t, \frac{v_2 \sin t - v_3 \cos t}{1 + v_1}, \frac{v_2 \cos t + v_3 \sin t}{1 + v_1} \right) ,$$

where $t \in [0, 2\pi]$. Let us assume that $r_1 = \frac{1}{2}$ (this assumption will be useful in the next Chapter). Then, the resulting $r_v$ is:

$$r_v = \left( \frac{1}{2}, \frac{\sqrt{1 + 2v_1}}{2}, \frac{v_2 \sqrt{1 + 2v_1} - v_3}{2(1 + v_1)}, \frac{v_2 + v_3 \sqrt{1 + 2v_1}}{2(1 + v_1)} \right) . \tag{3.5}$$

Using this the required automorphisms for an arbitrary $v \in S^2 \setminus \{S, N\}$ are

$$Q_v \colon \mathbb{H} \to \mathbb{H}, \quad x \mapsto r_v x \bar{r}_v \ ,$$

$$\tilde{Q}_v \colon \mathbb{H} \to \mathbb{H}, \quad x \mapsto r_{-v} x \bar{r}_{-v} \ .$$

The transition function between the two trivializations is given by

$$\psi_1 \circ \psi_2^{-1} : S^2 \setminus \{S, N\} \times SO(2) \to S^2 \setminus \{S, N\} \times SO(2)$$

$$(v, \phi) \mapsto (v, \theta_v \circ \tilde{\theta}_v^{-1}(\phi)) \ .$$

Since $\psi_1$ and $\psi_2$ parametrizes an automorphism $\varphi \in p^{-1}$ as maps $V_i \to V_v$ and $V_{-i} \to V_v$ respectively, the function $\theta_v \circ \tilde{\theta}_v^{-1}$ gives a $V_i \to V_{-i}$ mapping for any $v$, which is the 'difference' between the bases $(Q_v(j), Q_v(k))$ and $(\tilde{Q}_v(j), \tilde{Q}_v(-k))$.

Since we cover the base space $S^2$ with two trivializing charts, it is enough to calculate the transition function on the equator, which is a deformation retract of the intersection of the charts. From this the whole fibre space can be recovered. The matrices of $Q_v$ ,$Q_{-v}$ and $Q_{-v}^{-1} \circ Q_v$ in case when $v_1 = 0$ are the followings:

$$[Q_v] = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_2 & -v_2 v_3 & -v_3^2 \\ v_3 & v_2^2 & v_2 v_3 \end{pmatrix} , \qquad [Q_{-v}] = \begin{pmatrix} 0 & v_3 & -v_2 \\ -v_2 & -v_2 v_3 & -v_3^2 \\ -v_3 & v_2^2 & v_2 v_3 \end{pmatrix} ,$$

$$[Q_{-v}^{-1} \circ Q_v] = [Q_{-v}]^T [Q_v] = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 - 2v_3^2 & 2v_2 v_3 \\ 0 & 2v_2 v_3 & 1 - 2v_2^2 \end{pmatrix} .$$

As a consequence, the effect of the composition $\theta_v \circ \tilde{\theta}_v^{-1} \colon SO(2) \to SO(2)$ on the equator of $S^2$ is given as

$$M \mapsto \begin{pmatrix} -(1 - 2v_3^2) & -2v_2 v_3 \\ 2v_2 v_3 & 1 - 2v_2^2 \end{pmatrix} \cdot M = \begin{pmatrix} -v_2^2 + v_3^2 & -2v_2 v_3 \\ 2v_2 v_3 & -v_2^2 + v_3^2 \end{pmatrix} \cdot M \ .$$

This is the function with which the two trivial $SO(2) \times D^2$ bundles are glued together at the boundaries of the discs.

As mentioned before, the second column vector of a matrix in $SO(2)$ is determined by the first. We parametrize the unit length $v_1$ in the equator and $v_2$ in its normal plane (which can be thought as the tangent plane of $S^2$ at $v_1$) with their arcs, i.e. $v_{12} = \cos u$, $v_{13} = \sin u$, $v_{22} = \cos t$, $v_{23} = \sin t$. Then, the transition function between the two trivializing neighborhoods as an $S^1 \to S^1$ mapping can be written as

$$\psi_{12}(v_{12}, v_{13}, v_{22}, v_{23}) = \psi_{12}(\cos u, \sin u, \cos t, \sin t) =$$

$$= ((-\sin^2 u + \cos^2 u)\cos t - 2\cos u \sin u \sin t, 2\cos u \sin u \cos t + (-\sin^2 u + \cos^2 u)\sin t)$$

$$= (\cos 2u \cos t - \sin 2u \sin t, \sin 2u \cos t + \cos 2u \sin t) = (\cos(2u + t), \sin(2u + t)) \ .$$

This means, that as we move around the equator of $S^2$, the fibers $S^1$ are glued together with a constantly increasing twist. In particular, during the full path around the equator the fibers are twisted twice. This effect is just the twice of the generator of $\pi_1(S^1)$, the group which classifies the $SO(2)$-, or equivalently the $S^1$-bundles over $S^2$.

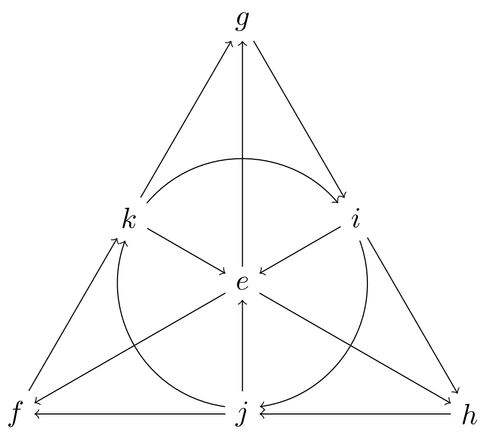



Figure 3.1: A simple mnemonic for the products of the basis vectors of $\mathbb{O}$.

## 3.3 Cayley numbers

The doubling of the algebra of quaternions leads to a new, 8 dimensional algebra over the reals.

**Definition 3.15.** The algebra $\mathbb{O} = \mathbb{H}^2$ is the *Cayley algebra*, and its elements are called *octonions* or *Cayley numbers.*

By definition every octonion is of the form $\xi = a + be$, where $a$ and $b$ are quaternions. The basis of $\mathbb{O}$ consists of 1 and seven elements

$$i,\ j,\ k,\ e,\ f = ie,\ g = je,\ h = ke\ .$$

The square of each of these elements is $-1$, and they are orthogonal to 1. A convenient mnemonic for remembering the products of the basis octonions is given by Figure 3.1, which represents the multiplication table of the basis elements [20]. This diagram with seven points and seven lines (the circle through $i$, $j$, and $k$ is considered a line) is called the Fano plane. The lines are oriented and the seven points correspond to the seven standard basis elements of $\mathbb{O}'$. Each pair of distinct points lies on a unique line and each line runs through exactly three points.

Let $(a, b, c)$ be an ordered triple of points lying on a given line with the order specified by the direction of the arrow. Then multiplication is given by

$$ab = c, \qquad ba = -c$$

together with cyclic permutations. These rules together with

- 1 is the multiplicative identity,
- $a^2 = -1$ for each point in the diagram

completely defines the multiplicative structure of the octonions. Each of the seven lines together with 1 generates a subalgebra of $\mathbb{O}$ isomorphic to the quaternions $\mathbb{H}$.

If follows from the identities in (3.1) that the algebra $\mathbb{O}$ is not associative because $\mathbb{H}$ is not commutative. Nevertheless, it satisfies a weaker form of associaticity.

**Definition 3.16.** An algebra $\mathcal{A}$ is *alternative* if

$$(\xi\eta)\eta = \xi(\eta\eta), \qquad \xi(\xi\eta) = (\xi\xi)\eta$$

for all $\xi, \eta \in \mathcal{A}$.

**Lemma 3.17.** *The algebra $\mathbb{O}$ is alternative.*

*Proof.* Let $\xi = a + be$, $\eta = u + ve$. According to the rules of multiplication in a doubled algebra

$$\xi\eta = (au - \overline{v}b + (b\overline{u} + va))e \;,$$

$$(\xi\eta)\eta = ((au - \overline{v}b)u - \overline{v}(b\overline{u} + va)) + ((b\overline{u} + va)\overline{u} + v(au - \overline{v}b))e \;,$$

$$\eta\eta = (u^2 - \overline{v}v) - (v\overline{u} + vu)e \;,$$

$$\xi(\eta\eta) = (a(u^2 - \overline{v}v) - \overline{(\overline{v}u + vu)}b) + (b\overline{(\overline{u}^2 - \overline{v}v)} + (v\overline{u} + vu)a)e \;.$$

The numbers $\overline{v}v = v\overline{v}$ and $u + \overline{u}$ are real and so commute with any quaternions. Thus,

$$\begin{aligned} a(u^2 - \overline{v}v) - \overline{(\overline{v}u + vu)}b &= au^2 - a\overline{v}v - (u + \overline{u})\overline{v}b \\ &= au^2 - \overline{v}va - \overline{v}b(u + \overline{u}) \\ &= (au - \overline{v}b)u - \overline{v}(b\overline{u} + va) \;, \end{aligned}$$

$$\begin{aligned} b\overline{(\overline{u}^2 - \overline{v}v)} + (v\overline{u} + vu)a &= b\overline{u}^2 - b\overline{v}v + v(\overline{u} + u)a \\ &= b\overline{u}^2 - v\overline{v}b + va(\overline{u} + u) \\ &= (b\overline{u} + va)\overline{u} + v(au - v\bar{b}) \;. \end{aligned}$$

This means that $(\xi\eta)\eta = \xi(\eta\eta)$. The equation $\xi(\xi\eta) = (\xi\xi)\eta$ can be prooved similarly. $\square$

Due to Lemma 3.3 $\mathbb{O}$ is a metric algebra. Next we show that $\mathbb{O}$ is a normed algebra as well.

**Lemma 3.18.** *The algebra $\mathbb{O}$ is a normed algebra with the norm generated by the metric, hence it is a division algebra.*

*Proof.* Let $\xi = a + be$, $\eta = u + ve$ be two octonions. Then

$$|\xi\eta|^2 = |au - v\bar{b}|^2 + |b\bar{u} + va|^2 = (au - \bar{v}b)(\bar{u}\bar{a} - \bar{b}v) + (b\bar{u} + va)(u\bar{b} + \bar{a}\bar{v}) \;,$$

$$|\xi|^2|\eta|^2 = (a\bar{a} + b\bar{b})(u\bar{u} + v\bar{v}) \;.$$

Suppose $v = \lambda + v'$, where $\lambda \in \mathbb{R}$ and $v' \in \mathbb{H}'$, and hence $\bar{v'} = -v'$. From this it follows that

$$|\xi\eta|^2 - |\xi|^2|\eta|^2 = \lambda(au\bar{b} + b\bar{u}\bar{a} - b\bar{u}\bar{a} - au\bar{b}) - (au\bar{b} + b\bar{u}\bar{a})v' + v'(au\bar{b} + b\bar{u}\bar{a}) = 0$$

because $au\bar{b} + b\bar{u}\bar{a}$ is real and therefore commutes with $v'$. $\square$

As a consequence of Lemma 3.6 $\mathbb{O}$ is an eight dimensional alternative division algebra.

# Chapter 4

# The exceptional Lie group $G_2$

In this chapter we investigate the group $G_2$ from two (and a half) aspects. First, it is the automorphism group of a specific algebra. As a consequence, it is a total space of a principal bundle. Finally, it is also the isotropy group of a generic 3-form.

## 4.1 $G_2$ as the automorphism group of the octonions

As mentioned in Section 3.2, the set of all automorphism of an algebra is a Lie group, and the Lie algebra of this Lie group is the set of all derivations of the algebra. Because $\mathbb{O}$ is an 8 dimensional, normed algebra according to Corollary 3.12

$$\operatorname{Aut}\mathbb{O} \subset O(7) \ ,$$

and hence

$$\operatorname{Der}\mathbb{O} \subset \mathfrak{so}(7) \ .$$

**Proposition 4.1** ([19], Lecture 14)**.** *The dimension of the Lie group* $\operatorname{Aut}\mathbb{O}$ *and its corresponding Lie algebra* $\operatorname{Der}\mathbb{O}$ *is 14. In addition,* $\operatorname{Aut}\mathbb{O}$ *is connected and simply connected.*

It can be shown as well, that $\dim O(7) = \dim \mathfrak{so}(7) = 21$, so the inclusions above are proper. With some calculations the root system and the Dynkin diagram of the Lie algebra $\operatorname{Der}\mathbb{O}$ can be derived. The result is shown in Figure 4.1. This means that $\operatorname{Der}\mathbb{O}$ is the exceptional simple Lie algebra $\mathfrak{g}_2$ and $\operatorname{Aut}\mathbb{O}$ is the compact form of the exceptional Lie group $G_2$.

### 4.1.1 Some useful identities

To perform calculations in the group $G_2$ further investigations about the structure of the Cayley algebra should be carried out.

**Definition 4.2.** The *associator* of three elements (in an arbitrary algebra) is the trilinear map defined by

$$[a, b, c] = (ab)c - a(bc) \ .$$

The alternativity of the algebra means precisly that for all $a, b$

$$[a, a, b] = 0, \quad [a, b, b] = 0 \ .$$

Both of these identities together imply that for an alternative algebra the associator is totally skew-symmetric (or alternative). That is,

$$[a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}] = sgn(\sigma)[a_1, a_2, a_3]$$

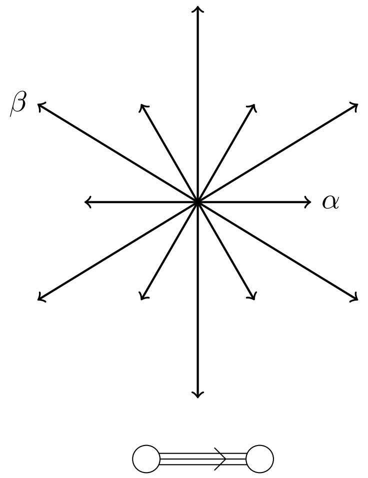



Figure 4.1: The root system and the Dynkin diagram of $G_2$.

for any permutation $\sigma$. From this it follows that

$$[a, b, a] = 0 ,$$

which means that

$$(ab)a = a(ba) .$$

This equation is known as the *identity of elasticity* (or *flexibility*).

A useful trick is the method of polarization, when instead of e.g. $b$ one writes $x + y$. By polarizing the three identities of the associator above one gets the following useful identities:

$$ax \cdot y + ay \cdot x = a \cdot xy + a \cdot yx \tag{4.1}$$

$$ax \cdot y + xa \cdot y = a \cdot xy + x \cdot ay \tag{4.2}$$

$$ax \cdot y + yx \cdot a = a \cdot xy + y \cdot xa \tag{4.3}$$

where instead of parantheses, dots denote the order of multiplication.

**Lemma 4.3.** *The following relations are true between the octonion multiplication and the scalar product:*

$$ax \cdot \overline{y} + ay \cdot \overline{x} = 2\langle x, y\rangle a , \tag{4.4}$$

$$\langle ux, vy\rangle + \langle vx, uy\rangle = 2\langle u, v\rangle\langle x, y\rangle . \tag{4.5}$$

*Proof.* Since $\mathbb{O}$ is a metric algebra any elements $a, b \in \mathbb{O}$ with $b = \lambda + b'$, $\lambda \in \mathbb{R}, b' \perp 1$ satisfies $(ab)(\lambda + b') = a \cdot b(\lambda + b')$, and therefore $ab \cdot b' = a \cdot bb'$. Hence $ab \cdot \lambda - ab \cdot b' = a \cdot b\lambda - a \cdot bb'$. As a consequence

$$ab \cdot \bar{b} = a \cdot b\bar{b} = a\langle b, b\rangle .$$

Polirizing this identity leads to (4.4).

The algebra $\mathbb{O}$ is in addition a normed algebra with the norm induced by the metric. Consequently, $\langle ab, ab\rangle = \langle a, a\rangle\langle b, b\rangle$ for all $a, b \in \mathbb{O}$. Polarizing this first by $b = x + y$ and then by $a = u + v$ we obtain (4.5). □

A very important group identities are given by the next Lemma.

**Lemma 4.4** (Moufang identities)**.** *In every alternative algebra there are identities*

$$c \cdot aba = (ca \cdot b)a \ , \tag{4.6}$$
$$aba \cdot c = a(b \cdot ac) \ , \tag{4.7}$$
$$a \cdot bc \cdot a = ab \cdot ca \ . \tag{4.8}$$

*Proof.* Due to the identities of alternativity and elasticity one has

$$x^2c \cdot x = (x \cdot xc)x = x(xc \cdot x) = x(x \cdot cx) = x^2 \cdot cx$$

and due to the skew-symmetry of the associator this leads to

$$cx^2 \cdot x = c \cdot x^3 \ ,$$

and thus

$$c \cdot x^3 = (cx \cdot x)x \ .$$

Polarizing this identity with $x = a + b$ and grouping similar terms we obtain

$$c{\cdot}a^2b{+}c{\cdot}ba^2{+}x{\cdot}aba{+}c{\cdot}bab{+}c{\cdot}ab^2{+}c{\cdot}b^2a = ca^2{\cdot}b{+}cb{\cdot}a^2{+}(ca{\cdot}b)a{+}(cb{\cdot}a)b{+}ca{\cdot}b^2{+}cb^2{\cdot}a \ .$$

Taking into account the skew-symmetry of the associator the sums of the first two terms on each side are equal. For the same reason so are the sums of the last wo terms. Therefore,

$$c \cdot aba + c \cdot bab = (ca \cdot b)a + (cb \cdot a)b \ .$$

Now we replace $b$ with $\lambda b$, where $\lambda \in \mathbb{R}$, then we divide both sides by $\lambda$ and take $\lambda = 0$. This results in

$$c \cdot aba = (ca \cdot b)a \ ,$$

which is 4.6 and 4.7 can be prooved similarly.

Moreover, (4.8) can be prooved by using again the skew-symmetry of the associator, i.e.

$$\begin{aligned} ab \cdot ca - a \cdot bc \cdot a &= ab \cdot ca - (ab \cdot x)a + (ab \cdot c - a \cdot bc)a \\ &= (c \cdot ab)a - c \cdot aba + (ca \cdot b - c \cdot ab)a \\ &= (ca \cdot b)a - c \cdot aba = 0 \ . \end{aligned}$$

□

### 4.1.2 The subgroup $SU(3)$

Consider the subset of the vector space $\mathbb{O}'$ consisting of elements $\xi$, such that $|\xi| = 1$. This set is a 6-dimensional sphere, which is denoted by $S^6$. An automorphism $\Phi\colon \mathbb{O} \to \mathbb{O}$ sends the elements $i, j$ and $e$ to elements $\xi = \Phi i, \eta = \Phi j$ and $\zeta = \Phi e$ in $S^6$ such that $\eta$ is orthogonal to $\xi$ and $\zeta$ is orthogonal to $\xi, \eta$ and $\xi\eta$. The next theorem shows, that these conditions are not only necessary but also sufficient for the existence of the automorphism $\Phi$.

**Theorem 4.5.** *For any elements $\xi, \eta, \zeta \in S^6$ such that*

*(a) $\eta$ is orthogonal to $\xi$*

*(b) $\zeta$ is orthogonal to $\xi, \eta$ and $\xi\eta$*

*there is a unique automorphism $\Phi\colon \mathbb{O} \to \mathbb{O}$ for which*

$$\xi = \Phi i, \qquad \eta = \Phi j, \qquad \zeta = \Phi e \ .$$

For the proof of this theorem we will follow [19]. The forthcoming lemmas will be necessary.

**Lemma 4.6.** *$\xi \in S^6$ if and only if $\xi^2 = -1$.*

*Proof.* If $\xi \in \mathbb{O}'$, then $\bar{\xi} = -\xi$ and therefore $\xi^2 = -|\xi|^2$. Hence if $|\xi| = 1$, then $\xi^2 = -1$. Conversely if $\xi^2 = -1$, then $|\xi| = \xi\bar{\xi} = 1$, that is $\xi(-\bar{\xi}) = -1 = \xi\xi$. Hence $\bar{\xi} = -\xi$, i.e. $\bar{\xi} \in \mathbb{O}'$, and since $|\xi| = 1$, we have $\xi \in S^6$. ☐

Let $\mathcal{H}$ be a unital algebra (and therefore closed under conjugation) subalgebra of $\mathbb{O}$ other than $\mathbb{O}$ and let $\xi$ be an octave in $S^6$ orthogonal to $\mathcal{H}$.

**Lemma 4.7.** *For any element $b \in \mathcal{H}$ the octonion $b\xi$ is orthogonal to $\mathcal{H}$.*

*Proof.* Applying (4.5) to $u = 1, v = b, x = \xi$ and $y = a$, where $a \in \mathcal{H}$, one has

$$\langle \xi, ba\rangle + \langle b\xi, a\rangle = 2\langle 1, b\rangle\langle \xi, a\rangle \ .$$

Since $ba \in \mathcal{H}$, $\langle \xi, ba\rangle = 0$ and by assumption $\langle \xi, a\rangle = 0$. Therefore, $\langle b\xi, a\rangle = 0$ for any $a \in \mathcal{H}$. ☐

In particular, $b\xi \perp 1$ (since $1 \in \mathcal{H}$), so that $\overline{b\xi} = -b\xi$.

**Lemma 4.8.** *For all $a, b \in \mathcal{H}$,*

$$\begin{aligned} a\xi \cdot b &= a\bar{b} \cdot \xi \ , \\ a \cdot b\xi &= ba \cdot \xi \ , \\ a\xi \cdot b\xi &= -\bar{b}a \ . \end{aligned} \tag{4.9}$$

*Proof.* First, applying (4.4) to $x = \xi, y = \bar{b}$ and taking into account that $\xi \perp b$ and therefore $\xi \perp \bar{b}$ we obtain the equation

$$a\xi \cdot b + a\bar{b} \cdot \xi = 2\langle \xi, \bar{b}\rangle = 0 \ ,$$

which is equivalent to the first identity.

Second, using again (4.4) to $a = 1, x = a, y = \bar{b}\xi = -b\xi$ yields

$$-a \cdot b\xi + b\xi \cdot \bar{a} = -2a\langle a, b\xi\rangle \ ,$$

which is zero by Lemma 4.7. Hence

$$a \cdot b\xi = b\xi \cdot \bar{a} \ .$$

Applying the first identity to the right side $b\xi \cdot \bar{a} = ba \cdot \xi$, and thus

$$a \cdot b\xi = ba \cdot \xi \ .$$

Finally, with $b \in \mathbb{R}$ the third identity becomes $a\xi \cdot b\xi = -ba$ and hence reduces to the first identity. It suffices therefore to prove that identity only for $b \perp 1$. In this case using (4.5) with $x = \xi$ and $y = \overline{b\xi} = -b\xi$ leads to

$$a\xi \cdot b\xi + (a \cdot b\xi)\xi = -2\langle \xi, b\xi \rangle a = 0 \ ,$$

because due to (4.5) for $u = 1, v = b, x = y = \xi$, $(\xi, b\xi) = (1, b)(\xi, \xi) = 0$. Hence, using the preceding identities

$$a\xi \cdot b\xi = -(a \cdot b\xi)\xi = -(ba \cdot \xi)\xi = -(ba) \cdot \xi\xi = ba = -\bar{b}a \ .$$

□

**Lemma 4.9.** *$\mathcal{H}$ is an associative algebra.*

*Proof.* If $a, b, c \in \mathcal{H}$, then applying (4.4) to $a = b\xi, x = c$ and $y = \bar{a}\xi$ we get

$$(b\xi \cdot \bar{c}) \cdot \overline{\bar{a}\xi} + (b\xi \cdot \bar{a}\xi)c = 2\langle \bar{c}, \bar{a}\xi \rangle \cdot b\xi = 0 \ .$$

Using the identities of Lemma 4.8 we also have

$$(b\xi \cdot \bar{a}\xi)c = -(ab)c \ ,$$

$$(b\xi \cdot \bar{c}) \cdot \overline{\bar{a}\xi} = -(bc \cdot \xi) \cdot \bar{a}\xi = a(bc) \ ,$$

i.e. for any $a, b, c \in \mathcal{H}$

$$a(bc) = (ab)c \ .$$

□

*Proof of Theorem 4.5.* It follows from the fact $\xi, \eta \in S^6$, that $\xi^2 = \eta^2 = -1$. Moreover, $\xi\eta = -\eta\xi$, since $\xi \perp \eta$. Therefore,

$$\overline{\xi\eta} = \bar{\eta}\bar{\xi} = \eta\xi = -\xi\eta \ .$$

This means that $\xi\eta \in \mathbb{O}'$, and because $|\xi\eta| = |\xi||\eta| = 1$, one has $\xi\eta \in S^6$. Consequently $(\xi\eta)^2 = -1$. Using the identity of alternativity $\xi(\xi\eta) = (\xi\xi)\eta = -\eta$ and $(\xi\eta)\eta = \xi(\eta\eta) = -\xi$. Adopting (4.5) to $u = v = \xi, x = \eta$ and $y = 1$ yields to

$$\langle \xi\eta, \xi \rangle = \langle \xi, \xi \rangle \langle \eta, 1 \rangle = 0 \ ,$$

from which it follows that

$$(\xi\eta)\xi = -\xi(\xi\eta) = -(\xi\xi)\eta = \eta \ .$$

Similarly, it can be shown that $\eta(\xi\eta) = \xi$ and this means that multiplying any number of the elements $\xi$ and $\eta$ in any order only the elements $\pm 1, \pm\xi, \pm\eta$ and $\pm\xi\eta$ can be obtained. That is, the elements of the form

$$a + b\xi + c\eta + d\xi\eta, \quad a, b, c, d \in \mathbb{R}$$

constitute a 4 dimensional subalgebra $\mathcal{H}$ of $\mathbb{O}$, which is an associative subalgebra due to Lemma 4.9. That is, the correspondences

$$1 \mapsto 1, \quad i \mapsto \xi, \quad j \mapsto \eta, \quad k \mapsto \xi\eta$$

define an isomorphism of the algebra of quaternions $\mathbb{H}$ onto the algebra $\mathcal{H}$.

Moreover because $\zeta$ is by assumption orthogonal to the elements $1, \xi, \eta$ and $\xi\eta$, it is orthogonal to the entire algebra $\mathcal{H}$, and therefore identities of Lemma 4.8 hold for it. From the second of these identities it follows that for the subalgebra generated by $\mathcal{H}$ and $\zeta$ the identity (4.4) is also true. Therefore it is possible to extend linearly the isomorphism $\mathbb{H} \to \mathcal{H}$ to a homomorphism of a subalgebra of $\mathbb{O}$ onto the subalgebra generated by $\mathcal{H}$ and $\zeta$ by sending $e$ to $\zeta$.

If a nonzero homomorphism of an unital division algebra is given, then it is a monomorphism, because if $\xi \neq 0$ were mapped to 0, $\xi^{-1}$ would not have a finite image. Therefore, the extended homomorphism $\Phi$ is a monomorphism of $\mathbb{O}$ into itself and in this way it is bijective, i.e. it is an automorphism of $\mathbb{O}$. This means that we have constructed an automorphism $\Phi : \mathbb{O} \to \mathbb{O}$ sending elements $i, j, e$ to $\xi, \eta, \zeta$ (and, of course, $k, f, g, h$ to $\xi\eta, \xi\zeta, \eta\zeta, (\xi\eta)\zeta$ respectively). □

From Theorem 4.5 it follows that the group $G_2 = \operatorname{Aut}\mathbb{O}$ acts transitively on $S^6$, i.e. the mapping $p\colon G_2 \to S^6$ defined by the formula $\Phi \mapsto \Phi i$ is surjective. Let us denote by $K$ the stabilizer (isotropy) group of $i$. This means, that $K = \{\Phi\colon \mathbb{O} \to \mathbb{O} \mid \Phi i = i\}$. Due to the standard theorem of transitive Lie group actions ([17], Theorem 9.24)

$$G_2/K \approx S^6 .$$

The subspace $V = \operatorname{Span}\{1, i\}^{\perp}$ of the algebra $\mathbb{O}$ is closed under the multiplication by $i$ and thus it can be considered as a vectorspace over the field $\mathbb{C}$ with basis $j, e, g$. The scalar product in $\mathbb{O}$ induces in $V$ a Hermitian scalar product with respect to which the basis $j, e, g$ is orthogonal. Any automorphism $\Phi\colon \mathbb{O} \to \mathbb{O}$ which leaves the element $i$ fixed, i.e. which is in the subgroup $K$, defines an operator $V \to V$ linear over $\mathbb{C}$. This operator preserves the scalar product, and therefore it is an unitary operator. Its determinant is 1 because $\operatorname{Aut}\mathbb{O} \subset SO(7)$ and therefore the group $K$ is identified with some subgroup of the group $SU(3)$. From Lemma 4.5 it also follows, that the group $K$ coincides with the entire group $SU(3)$. Thus it may be assumed, that $SU(3) \subset G_2$, with $G_2/SU(3) \approx S^6$.

**Corollary 4.10.** *Consider the evaluation mapping $p\colon G_2 \to S^6, \Phi \mapsto \Phi i$ defined above. This makes $G_2$ a locally trivial $SU(3)$-bundle over $S^6$.*

*Proof.* By identifying $SU(3)$ with $p^{-1}(i)$ we get an action

$$SU(3) \times G_2 \to G_2, (a, b) \to ab .$$

This $SU(3)$ action clearly carries fibers to fibers. Since $SU(3)$ is free and transitive on itself, it behaves the same way on $p^{-1}(i)$ and therefore on all of the fibers. □

### 4.1.3 The subgroup of inner automorphisms

In an associative division algebra, such as the quaternions over the reals, the mapping

$$q_r : x \mapsto rxr^{-1}$$

is an automorphism for any invertible element $r$. As mentioned in Section 3.2 these are called inner automorphism and in the case of $\mathbb{H}$ the set of inner automorphisms is the whole group $\operatorname{Aut}\mathbb{H}$. In a non-associative algebra not all the elements generate an inner automorphism. To define the inner automorphisms precisely in this case the following lemma is needed.

**Lemma 4.11.** *For any $r, x \in \mathbb{O}$*

$$(rx)r^{-1} = r(xr^{-1}) \ .$$

*Proof.* If the coordintes of $r$ in the standard basis are $(r_1, \ldots, r_8)$, then $r^{-1} = \frac{\bar{r}}{|r|^2} = \frac{2r_1 - r}{|r|^2}$. Therefore, using the identity of elasticity we have

$$(rx)r^{-1} = (rx)\frac{2r_1 - r}{|r|^2} = \frac{1}{|r|^2}((rx)2r_1 - (rx)r) = \frac{1}{|r|^2}(r(x2r_1) - r(xr)) = r(xr^{-1}) \ .$$

□

Now we can state the following important theorem.

**Theorem 4.12** ([16])**.** *A non-real octionion $r$ with coordinates $(r_1, \ldots, r_8)$ induces an inner automorphism of $\mathbb{O}$ if and only if $4r_1^2 = |r|^2$.*

*Proof.* From (4.8) for $a = r, b = xr^{-1}$ and $c = ryr$ it follows that

$$(rxr^{-1})(ryr \cdot r) = r(xr^{-1} \cdot ryr)r \ . \tag{4.10}$$

Similarly,

$$\overline{ryr} = \bar{r}\bar{y}\bar{r} = \bar{r}(\bar{y}(x^{-1}x))\bar{r} = \bar{r}((\bar{y}x^{-1})x)\bar{r} = (\bar{r} \cdot \bar{y}x^{-1})(x\bar{r})$$

and therefore

$$\begin{aligned} ryr &= \overline{(\bar{r} \cdot \bar{y}x^{-1})(x\bar{r})} = (\overline{x\bar{r}})(\overline{\bar{r} \cdot \bar{y}x^{-1}}) = (r\bar{x})(\overline{\bar{y}x^{-1}} \cdot r) \\ &= (r\bar{x})(\overline{x}^{-1}y \cdot r) = (r(|x|^2x^{-1}))\left(\frac{x}{|x|^2}y \cdot r\right) = (rx^{-1})(xy \cdot r) \ . \end{aligned}$$

Substituting this into (4.10) leads us to

$$\begin{aligned} (rxr^{-1})(ryr \cdot r) &= r(xr^{-1} \cdot (rx^{-1})(xy \cdot r))r = r((\underbrace{xr^{-1}}_{a} \cdot \underbrace{rx^{-1}}_{a^{-1}}) \cdot (xy \cdot r))r \\ &= r((xy \cdot r))r = r(xy)r^2 \ , \end{aligned}$$

i.e.

$$(rxr^{-1})(ryr^{-1} \cdot r^3) = r(xy)r^{-1} \cdot r^3 \tag{4.11}$$

for all $x, y, r \in \mathbb{O}$.

The mapping $q_r : x \mapsto rxr^{-1}$ is an automorphism if and only if

$$(rxr^{-1})(ryr^{-1}) = r(xy)r^{-1} \ .$$

Multiplying this with $r^3$ from the right we get

$$(rxr^{-1})(ryr^{-1}) \cdot r^3 = r(xy)r^{-1} \cdot r^3 \tag{4.12}$$

Comparing (4.11) to (4.12) we see that in order for $q_r$ to be an automorphism $r^3$ must be a scalar.

Using the fact that $\bar{r} = 2r_1 - r$, one has $|r|^2 = r\bar{r} = r(2r_1 - r) = 2rr_1 - r^2$ for all $r \in \mathbb{O}$. Multiplying with $r$ and applying the same equation again we get that

$$r^3 - 2r_1r^2 + |r|^2r = r^3 - 4r_1^2r + 2r_1|r|^2 + r|r|^2 = 0 \ ,$$

and thus

$$r^3 + 2r_1|r|^2 = r(4r_1^2 - |r|^2) \ .$$

Suppose $r^3$ is a scalar. Then each term on the left side is real and therefore either $r$ should be real, or $(4r_1^2 - |r|^2)$ should be zero. The latter case means that $4r_1^2 = |r|^2$. □

## 4.2 $G_2$ as an $SU(3)$-bundle over $S^6$

### 4.2.1 The transition function

Our aim is to determine the transition function between the charts of $S^6$ in a similar manner to the fibration $SO(3) \overset{SO(2)}{\longrightarrow} S^2$. The two charts on $S^6$ are $S^6 \setminus \{S\}$ and $S^6 \setminus \{N\}$. The preimage of $i$ is the set $p^{-1}(i) = \{(i, \eta, \zeta) : \eta \perp i, \zeta \perp \text{Span}\{i, \eta, i\eta\}\}$. As mentioned above this is isomorphic to $SU(3)$ and this isomorphism will be called $\theta_1$. From now on, elements in $p^{-1}(i)$ will be considered either as orthonormal vector triples in $V_i = T_i S^6$ or as operators that leave the vector $i$ fixed.

**Definition 4.13.** A *complex structure* on a vector space $V$ is a mapping $J : V \to V$, such that $J^2 = -\text{Id}_V$. That is, the effect of applying $J$ twice is the same as multiplication by $-1$.

A complex structure allows one to endow $V$ with the structure of a complex vector space. Complex scalar multiplication can be defined by

$$(x + Iy)v = xv + yJ(v) \; .$$

It is important not to confuse the imaginary unit $I$ with the octonion $i$. As mentioned earlier, there is a complex structure on $V_i$ given by $J_i(v) = iv$, using the octonion multiplication ($i$ is an octonion here). This is clearly a $V_i \to V_i$ mapping and $J_i^2(v) = i^2 v = -v$ for all $v \in V$. Thus, there is a $\theta_i \colon V_i \to \mathbb{C}^3$ isomorphism, that assigns to each operator $\Phi \in p^{-1}(i), \Phi : V_i \to V_i$ its matrix representation in the complex basis $\{j, e, g\}$.

In particular, the elements of the group $SU(3)$ are $3 \times 3$ matrices of the form $[v_1|v_2|v_3]$ consisting of complex orthogonal column vectors having unit length and where $v_3$ is the element in the subspace $\text{Span}_{\mathbb{C}}\{v_1, v_2\}^{\perp} \approx \mathbb{C}$ such that the determinant of the matrix is 1. As in the real case the third column is determined by the first two. For a particular $\Phi \in p^{-1}(i)$ the vectors $\eta = \Phi(j)$ and $\zeta = \Phi(e)$ are perpendicular to $i$ and complex orthogonal to each other. Thus, they can be thought as the first and second column of such a matrix and in this case the third column will be $\eta\zeta = \Phi(j)\Phi(e) = \Phi(je) = \Phi(g)$. If the coordinates of the vectors are $\eta = (0, y_2, \ldots, y_8)$, $\zeta = (0, z_2, \ldots, z_8)$ and $\eta\zeta = (0, u_2, \ldots, u_8)$, then since $\eta, \zeta, \eta\zeta \in V_i$, $y_2 = 0$, $z_2 = 0$ and $u_2 = 0$. The mapping $\theta_i$ is then the following:

$$\theta_i \colon p^{-1}(i) \to SU(3), \quad (i, \eta, \zeta) \mapsto \begin{pmatrix} y_3 + Iy_4 & z_3 + Iz_4 & u_3 + Iu_4 \\ y_5 + Iy_6 & z_5 + Iz_6 & u_5 + Iu_6 \\ y_7 + Iy_8 & z_7 + Iz_8 & u_7 + Iu_8 \end{pmatrix} ,$$

which is

$$(i, \eta, \zeta) \mapsto \begin{pmatrix} \langle \eta, j \rangle + I\langle \eta, k \rangle & \langle \zeta, j \rangle + I\langle \zeta, k \rangle & \langle \eta\zeta, j \rangle + I\langle \eta\zeta, k \rangle \\ \langle \eta, e \rangle + I\langle \eta, f \rangle & \langle \zeta, e \rangle + I\langle \zeta, f \rangle & \langle \eta\zeta, e \rangle + I\langle \eta\zeta, f \rangle \\ \langle \eta, g \rangle + I\langle \eta, h \rangle & \langle \zeta, g \rangle + I\langle \zeta, h \rangle & \langle \eta\zeta, g \rangle + I\langle \eta\zeta, h \rangle \end{pmatrix} ,$$

i.e. we represent $\eta, \zeta, \eta\zeta \in V_i$, the images of $j, e$ and $g$ in the complex basis $\{j, e, g\}$.

Similarly, $p^{-1}(\xi) = \{(\xi, \eta, \zeta) : \eta \perp \xi, \zeta \perp \text{Span}\{\xi, \eta, \xi\eta\}\}$ for any $\xi \in S^6$ . Let us denote by $V_\xi$ or $T_\xi S^6$ the tangent space (of orthogonal vectors) to $\xi$. Then any map $\varphi \in p^{-1}(\xi)$ carries $V_i$ to $V_\xi$. Again, there is a complex structure on $V_\xi$ denoted by $J_\xi$, which comes from octonion multiplication: $J_\xi(v) = \xi v$. By choosing a complex orthonormal basis in this subspace we give an identification $V_\xi \approx \mathbb{C}^3$.

**Definition 4.14** ([3])**.** An *almost complex manifold* is a real manifold $M$ equipped with a bundle map $J : TM \to TM$ satisfying the following properties:

1. $J(T_mM) = T_mM$ for all $m \in M$,
2. $J^2 = -1$.

In other words, an almost complex structure on a real manifold is a smoothly varying complex structure on each tangent space.

**Corollary 4.15.** *Because the complex structure on the vector space* $T_\xi S^6 = V_\xi$ *is given by* $J_\xi : V_\xi \to V_\xi, v \mapsto \xi v$, *is obviously smoothly varying, the manifold* $S^6$ *is an almost complex manifold with* $J : TS^6 \to TS^6, (\xi, v) \mapsto (\xi, J_\xi(v))$.

**Proposition 4.16.** *A rotation* $g : S^6 \to S^6$ *is an element of* $G_2$ *if and only if its push-forward* $g_* : TS^6 \to TS^6, (x, v) \mapsto (g(x), g(v))$ *is* $J$*-equivariant (where* $J$ *is considered as a* $\mathbb{Z}_4$*-action on* $TM$*), or in other words the following diagram is commutative:*

$$\begin{array}{ccc} TS^6 & \xrightarrow{\quad J \quad} & TS^6 \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ TS^6 & \xrightarrow{\quad J \quad} & TS^6. \end{array}$$

*Proof.* Because $G_2 \subset SO(7)$ any $g \in G_2$ preserves the scalar product. Therefore, $g(V_\xi) = V_{g(\xi)}$ and what we need to proove is that the following diagram commutes for all $\xi \in S^6$:

$$\begin{array}{ccc} T_\xi S^6 & \xrightarrow{\quad J_\xi \quad} & T_\xi S^6 \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ T_{g(\xi)} S^6 & \xrightarrow{\quad J_{g(\xi)} \quad} & T_{g(\xi)} S^6. \end{array}$$

Since $g \in \operatorname{Aut} \mathbb{O}$ we have that

$$g(J_\xi(\eta)) = g(\xi\eta) = g(\xi)g(\eta) = J_{g(\xi)}(g(\eta)) \ ,$$

for all $\xi \in S^6, \eta \in V_\xi$.

Conversely, assume $\xi \in S^6, \eta \in \mathbb{O}'$. Decompose $\eta$ to $\eta_1 + \eta_2$ where $\eta_1 \perp \xi$. Suppose $g_*$ commutes with $J$. Then

$$g(\xi\eta_1) = g(J_\xi(\eta_1)) = J_{g(\xi)}(g(\eta_1)) = g(\xi)g(\eta_1) \ ,$$

and obviously $g(\xi\eta_2) = g(\xi)g(\eta_2)$. Thus, $g(\xi\eta) = g(\xi)g(\eta)$. □

As a consequence, for a $\xi \in S^6 \setminus \{S\}$ any $\varphi \in p^{-1}(\xi)$ restricts to a mapping $V_i \to V_\xi$, which is complex linear, unitary and has determinant 1. We will choose a complex orthonormal basis in $V_\xi$ and write the images of $j, e$ and $g$ in this basis. That is, we choose particular identifications $V_i \approx \mathbb{C}^3, V_\xi \approx \mathbb{C}^3$ and we define $\theta_\xi \colon p^{-1}(\xi) \to SU(3)$ by assigning to each automorphism $\varphi \in p^{-1}(\xi)$ the matrix of the mapping $\varphi\colon \mathbb{C}^3 \to \mathbb{C}^3$. To find a basis in $V_\xi$ we will define a translating automorphism $Q_\xi$ such that $Q_\xi(i) = \xi$. Then for $a = Q_\xi(j), b = Q_\xi(e)$ and $c = Q_\xi(g)$, $\{a, b, c\}$ is a complex orthonormal basis in $V_\xi$ with respect to the complex structure $J_\xi(v) = \xi v$. Particularly,

$$\theta_\xi \colon p^{-1}(\xi) \to SU(3)$$

$$(\xi,\eta,\zeta)\mapsto\begin{pmatrix}\langle\eta,a\rangle+I\langle\eta,J_\xi(a)\rangle & \langle\zeta,a\rangle+I\langle\zeta,J_\xi(a)\rangle & \langle\eta\zeta,a\rangle+I\langle\eta\zeta,J_\xi(a)\rangle\\ \langle\eta,b\rangle+I\langle\eta,J_\xi(b)\rangle & \langle\zeta,b\rangle+I\langle\zeta,J_\xi(b)\rangle & \langle\eta\zeta,b\rangle+I\langle\eta\zeta,J_\xi(b)\rangle\\ \langle\eta,c\rangle+I\langle\eta,J_\xi(c)\rangle & \langle\zeta,c\rangle+I\langle\zeta,J_\xi(c)\rangle & \langle\eta\zeta,c\rangle+I\langle\eta\zeta,J_\xi(c)\rangle\end{pmatrix}.$$

Using this the trivializing map is given by

$$\psi_1\colon p^{-1}(S^6\setminus\{S\})\to S^6\setminus\{S\}\times SU(3),\quad \varphi\mapsto(\varphi(i),\theta_{\varphi(i)}(\varphi)).$$

Similarly, the preimage of $-i$ is diffeomorphic to $SU(3)$, and in this case the complex structure on $V_{-i}$ is given by $J_{-i}(v)=-iv$. Therefore, $\tilde\theta_{-i}$ is defined as

$$(-i,\eta,\zeta)\mapsto\begin{pmatrix}\langle\eta,j\rangle+I\langle\eta,-k\rangle & \langle\zeta,j\rangle+I\langle\zeta,-k\rangle & \langle\eta\zeta,j\rangle+I\langle\eta\zeta,-k\rangle\\ \langle\eta,e\rangle+I\langle\eta,-f\rangle & \langle\zeta,e\rangle+I\langle\zeta,-f\rangle & \langle\eta\zeta,e\rangle+I\langle\eta\zeta,-f\rangle\\ \langle\eta,g\rangle+I\langle\eta,-h\rangle & \langle\zeta,g\rangle+I\langle\zeta,-h\rangle & \langle\eta\zeta,g\rangle+I\langle\eta\zeta,-h\rangle\end{pmatrix}.$$

As we did in the previous case, for a general $\xi\in S^6\setminus\{N\}$ we will choose a translating automorphism $\tilde Q_\xi$ with the property that $\tilde Q_\xi(-i)=\xi$ and therefore $\tilde Q_\xi(j),\tilde Q_\xi(e),\tilde Q_\xi(g)\in V_\xi$ form a complex orthonormal bases. Then we define $\tilde\theta_\xi\colon p^{-1}(\xi)\to SU(3)$ by assigning to $\varphi\in p^{-1}(v)$ the matrix of the corresponding linear mapping from $V_{-i}$ onto $V_\xi$ written in the bases $\{j,e,g\}$ at $V_{-i}$ and $\{\tilde Q_\xi(j),\tilde Q_\xi(e),\tilde Q_\xi(g)\}$ at $V_\xi$. The second trivializing map is then given by

$$\psi_2\colon p^{-1}(S^2\setminus\{N\})\to S^2\setminus\{N\}\times SO(2),\quad \varphi\mapsto(\varphi(i),\tilde\theta_{\varphi(i)}(\varphi)).$$

To summarize, if $Q_\xi,\tilde Q_\xi\in G_2$ are known as functions of $\xi$ with the property that $Q_\xi(i)=\xi$ and $\tilde Q_\xi(-i)=\xi$, then an appropriate basis in $V_\xi$ is $a=Q_\xi(j),b=g_\xi(e),c=Q_\xi(g)$, which are the translations of the basis $j,e,g$ from $V_i$ in the case of the first chart. In the case of the second chart $\tilde Q_\xi$ translates $j,e,g$ from $V_{-i}$ to $V_\xi$. Thus, we need to find elements $Q_\xi\in G_2$ and $\tilde Q_\xi\in G_2$. Knowing the first one is enough, because then second is given since $Q_{-\xi}(-i)=Q_{-\xi}((-1)i)=Q_{-\xi}(-1)Q_{-\xi}(i)=-1(-\xi)=\xi$.

It is convenient to look for $Q_\xi$ in the form of an inner automophism generated by an $r\in\mathbb{O}$. The easiest is to look for a unit length octionon that induces $Q_\xi$. For a unit length octionion $r$ the conjugate of $i$ with $r$ is:

$$r i\bar r=(0,r_1^2+r_2^2-r_3^2-r_4^2-r_5^2-r_6^2-r_7^2-r_8^2,2(r_2r_3+r_1r_4),2(r_2r_4-r_1r_3),$$

$$2(r_2r_5+r_1r_6),2(r_2r_6-r_1r_5),2(r_2r_7-r_1r_8),2(r_1r_7+r_2r_8)).$$

Since $r_\xi i\bar r_\xi=\xi=(0,x_2,\dots,x_8)$ is needed, the following system of equations is to be sold:

$$\begin{aligned} r_1^2+r_2^2-r_3^2-r_4^2-r_5^2-r_6^2-r_7^2-r_8^2&=x_2\\ 2(r_2r_3+r_1r_4)&=x_3\\ 2(r_2r_4-r_1r_3)&=x_4\\ 2(r_2r_5+r_1r_6)&=x_5\\ 2(r_2r_6-r_1r_5)&=x_6\\ 2(r_2r_7-r_1r_8)&=x_7\\ 2(r_1r_7+r_2r_8)&=x_8\end{aligned}$$

From Theorem 4.12 it follows that $r_1=\frac{1}{2}$ is required. The general solution for an arbitrary $\xi\in S^6\setminus S$ of this system of equations is similar to that of (3.4) except the signs in the last pair of coordinates:

$$r_\xi = \left(\frac{1}{2}, \frac{\sqrt{1+2x_2}}{2}, \frac{x_3\sqrt{1+2x_2}-x_4}{2(1+x_2)}, \frac{x_3+x_4\sqrt{1+2x_2}}{2(1+x_2)},\right.$$
$$\left.\frac{x_5\sqrt{1+2x_2}-x_6}{2(1+x_2)}, \frac{x_5+x_6\sqrt{1+2x_2}}{2(1+x_2)}, \frac{x_7\sqrt{1+2x_2}+x_8}{2(1+x_2)}, \frac{-x_7+x_8\sqrt{1+2x_2}}{2(1+x_2)}\right) .$$

Again, we cover the base space $S^6$ with two trivializing charts and it is enough to calculate the transition function on the equator, which is a deformation retract of the intersection of the charts. From now on we assume that $\xi$ is in the equator of $S^6$, and thus $x_2 = 0$. In this case

$$r_\xi = \left(\frac{1}{2}, \frac{1}{2}, \frac{x_3-x_4}{2}, \frac{x_3+x_4}{2}, \frac{x_5-x_6}{2}, \frac{x_5+x_6}{2}, \frac{x_7+x_8}{2}, \frac{-x_7+x_8}{2}\right) .$$

Dut to the fact that $i\xi = (0, 0, -x_4, x_3, -x_6, x_5, x_8, -x_7)$, we have

$$r_\xi = \frac{1}{2} + \frac{i}{2} + \frac{\xi + i\xi}{2} = \frac{(1+i)(1+\xi)}{2} .$$

It is easy to check that $r_\xi$ is really a solution, because in this case due to the identity of elasticity and Lemma 4.11 we may perform the multiplication in arbitrary order:

$$\frac{(1+i)(1+\xi)}{2} \cdot i \cdot \overline{\frac{(1+i)(1+\xi)}{2}} = \frac{1}{4}(1+i)((1+\xi)i(1-\xi))(1-i)$$
$$= \frac{1}{4}(1+i)(i+\xi i - i\xi - \xi i\xi)(1-i) = \frac{1}{4}(1+i)(i+2\xi i + i\xi^2)(1-i) = \frac{1}{4}(1+i)2\xi i(1-i)$$
$$= \frac{1}{4}(2\xi i + 2i\xi i - 2\xi i^2 - 2i\xi i^2) = \frac{1}{4}(2\xi i - 2i^2\xi + 2\xi + 2i\xi) = \frac{1}{4}(2\xi i + 4\xi - 2\xi i) = \frac{4\xi}{4} = \xi .$$

Consequently, the required automorphisms for an arbitrary $\xi \in S^6 \setminus \{S, N\}$ are

$$Q_\xi \colon \mathbb{O} \to \mathbb{O}, \quad x \mapsto r_\xi x \bar{r}_\xi ,$$
$$\tilde{Q}_\xi \colon \mathbb{O} \to \mathbb{O}, \quad x \mapsto r_{-\xi} x \bar{r}_{-\xi} .$$

Once again, the trivializing maps and the transition function between the two trivializations are

$$\psi_1 \colon p^{-1}(S^6 \setminus \{S\}) \to S^6 \setminus \{S\} \times SU(3), \quad \varphi \mapsto (\varphi(i), \theta_{\varphi(i)}) ,$$
$$\psi_2 \colon p^{-1}(S^6 \setminus \{N\}) \to S^6 \setminus \{N\} \times SU(3), \quad \varphi \mapsto (\varphi(i), \tilde{\theta}_{\varphi(i)}) ,$$
$$\psi_1 \circ \psi_2^{-1} \colon S^6 \setminus \{S, N\} \times SU(3) \to S^6 \setminus \{S, N\} \times SU(3) ,$$
$$(\xi, \phi) \mapsto (\xi, \theta_\xi \circ \tilde{\theta}_\xi^{-1}(\phi)) .$$

As it was discussed above, the meaning of $\psi_1$ is the following:

$$(\xi, \eta, \zeta) \mapsto \begin{pmatrix} \langle\eta, Q_\xi j\rangle + I\langle\eta, Q_\xi k\rangle & \langle\zeta, Q_\xi j\rangle + I\langle\zeta, Q_\xi k\rangle & \langle\eta\zeta, Q_\xi j\rangle + I\langle\eta\zeta, Q_\xi k\rangle \\ \langle\eta, Q_\xi e\rangle + I\langle\eta, Q_\xi f\rangle & \langle\zeta, Q_\xi e\rangle + I\langle\zeta, Q_\xi f\rangle & \langle\eta\zeta, Q_\xi e\rangle + I\langle\eta\zeta, Q_\xi f\rangle \\ \langle\eta, Q_\xi g\rangle + I\langle\eta, Q_\xi h\rangle & \langle\zeta, Q_\xi g\rangle + I\langle\zeta, Q_\xi h\rangle & \langle\eta\zeta, Q_\xi g\rangle + I\langle\eta\zeta, Q_\xi h\rangle \end{pmatrix}$$

Similarly, $\psi_2$ is

$$(\xi, \eta, \zeta) \mapsto \begin{pmatrix} \langle\eta, \tilde{Q}_\xi j\rangle + I\langle\eta, \tilde{Q}_\xi k\rangle & \langle\zeta, \tilde{Q}_\xi j\rangle + I\langle\zeta, \tilde{Q}_\xi k\rangle & \langle\eta\zeta, \tilde{Q}_\xi j\rangle + I\langle\eta\zeta, \tilde{Q}_\xi k\rangle \\ \langle\eta, \tilde{Q}_\xi e\rangle + I\langle\eta, \tilde{Q}_\xi f\rangle & \langle\zeta, g_{-\xi} e\rangle + I\langle\zeta, \tilde{Q}_\xi f\rangle & \langle\eta\zeta, \tilde{Q}_\xi e\rangle + I\langle\eta\zeta, \tilde{Q}_\xi f\rangle \\ \langle\eta, \tilde{Q}_\xi g\rangle + I\langle\eta, \tilde{Q}_\xi h\rangle & \langle\zeta, g_{-\xi} g\rangle + I\langle\zeta, \tilde{Q}_\xi h\rangle & \langle\eta\zeta, \tilde{Q}_\xi g\rangle + I\langle\eta\zeta, \tilde{Q}_\xi h\rangle \end{pmatrix}$$

The mapping $Q_\xi(v) = r_\xi v \bar{r}_\xi$ is linear in $v$, because $\mathbb{O}$ is distributive and scalars commute with everything. Due to the construction $Q_\xi(x)$ maps the subspace $V_i$ to $V_\xi$ isomorphically.

**Lemma 4.17.** *If $v, \xi \in V_i$, then*

$$Q_\xi(v) = \frac{1}{2}((-1 + i + \xi + i\xi)v + \langle v, \xi + i\xi\rangle(1 + i + \xi + i\xi))$$

*Proof.* To compute $Q_\xi(v)$ 4 groups of identities will be necessary.

1. According to the definition of the scalar product in $\mathbb{O}$ and (4.4)

$$\begin{aligned} v \cdot i\xi &= -\overline{i\xi} \cdot \overline{v} + 2\langle v, \overline{i\xi}\rangle = \xi i \cdot v + 2\langle v, \xi i\rangle = -i\xi \cdot v - 2\langle v, i\xi\rangle\ , \\ iv \cdot \xi &= -i\overline{\xi} \cdot \overline{v} + 2\langle v, \overline{\xi}\rangle i = -i\xi \cdot v - 2\langle v, \xi\rangle i\ , \\ \xi v \cdot i &= -\xi\overline{i} \cdot \overline{v} + 2\underbrace{\langle i, \overline{v}\rangle}_{0}\xi = i\xi \cdot v\ . \end{aligned}$$

Therefore

$$i\xi \cdot v - \xi v \cdot i - iv \cdot \xi - v \cdot i\xi = 2i\xi \cdot v + 2\langle v, \xi\rangle i + 2\langle v, i\xi\rangle\ . \tag{4.13}$$

2. Similarly,

$$\begin{aligned} iv \cdot i\xi &= -(i \cdot \overline{i\xi})\overline{v} + 2\langle i\xi, \overline{v}\rangle i = (i \cdot \xi i)v + 2\langle \xi i, v\rangle i = \xi v + 2\langle \xi i, v\rangle i\ , \\ (i\xi \cdot v)i &= -(i\xi \cdot \overline{i})\overline{v} + 2\underbrace{\langle v, \overline{i}\rangle}_{0} i\xi = -(i\xi i)v = (ii\xi)v = -\xi v\ . \end{aligned}$$

Summing over the two equations this leads to

$$iv \cdot i\xi + (i\xi \cdot v)i = \xi v + 2\langle \xi i, v\rangle i - \xi v = 2\langle \xi i, v\rangle i\ . \tag{4.14}$$

3. With essentially the same tricks one obtains

$$\begin{aligned} (i\xi \cdot v)\xi &= -(i\xi \cdot \overline{\xi})\overline{v} + 2\langle v, \overline{\xi}\rangle i\xi = iv - 2\langle v, \xi\rangle i\xi\ , \\ \xi v \cdot i\xi &= -\xi\overline{i\xi} \cdot \overline{v} + 2\langle \overline{v}, i\xi\rangle\xi = -(\xi \cdot \overline{xi})\overline{v} - 2\langle v, i\xi\rangle\xi = -iv - 2\langle v, i\xi\rangle\xi\ . \end{aligned}$$

Therefore

$$(i\xi \cdot v)\xi + \xi v \cdot i\xi = iv - 2\langle v, \xi\rangle i\xi - iv - 2\langle v, i\xi\rangle\xi = -2\langle v, \xi\rangle i\xi - 2\langle v, i\xi\rangle\xi\ . \tag{4.15}$$

4. Once again,

$$\xi v\xi = -\xi\overline{\xi} \cdot \overline{v} + 2\langle \xi, \overline{v}\rangle\xi = v - 2\langle \xi, v\rangle\xi\ , \tag{4.16}$$

$$i\xi \cdot v \cdot i\xi = -(i\xi)\overline{i\xi} \cdot \overline{v} + 2\langle i\xi, \overline{v}\rangle i\xi = v - 2\langle i\xi, v\rangle i\xi\ . \tag{4.17}$$

Putting these together,

$$\begin{aligned} Q_\xi(v) &= r_\xi v \bar{r}_\xi = \tfrac{1}{4}(1 + i + \xi + i\xi)v(1 - i - \xi - i\xi) \\ &= \tfrac{1}{4}(v + iv + \xi v + i\xi \cdot v)(1 - i - \xi - i\xi) \\ &= \tfrac{1}{4}(v + iv + \xi v + i\xi \cdot v - vi - ivi - \xi v \cdot i - (i\xi \cdot v)i \\ &\quad -v\xi - iv \cdot \xi - \xi v\xi - (i\xi \cdot v)\xi - v \cdot i\xi - iv \cdot i\xi - \xi v \cdot i\xi - i\xi \cdot v \cdot i\xi) \\ &= \tfrac{1}{4}(2iv + \xi v - v\xi + (i\xi \cdot v - \xi v \cdot i - iv \cdot \xi - v \cdot i\xi) \\ &\quad -((i\xi \cdot v)i + iv \cdot i\xi) - ((i\xi \cdot v)\xi + \xi v \cdot i\xi) - \xi v\xi - i\xi \cdot v \cdot i\xi) \\ &= \tfrac{1}{4}(2iv - 2v + 2i\xi \cdot v + \xi v - v\xi + 2\langle v, \xi\rangle i + 2\langle v, i\xi\rangle \\ &\quad -2\langle \xi i, v\rangle i + 2\langle v, \xi\rangle i\xi + 2\langle v, i\xi\rangle\xi + 2\langle \xi, v\rangle\xi + 2\langle i\xi, v\rangle i\xi) \\ &= \tfrac{1}{4}(2iv - 2v + 2i\xi \cdot v + 2\xi v + 2\langle v, \xi\rangle + 2\langle v, \xi\rangle i + 2\langle v, i\xi\rangle \\ &\quad -2\langle \xi i, v\rangle i + 2\langle v, \xi\rangle i\xi + 2\langle v, i\xi\rangle\xi + 2\langle \xi, v\rangle\xi + 2\langle i\xi, v\rangle i\xi) \\ &= \tfrac{1}{2}(iv - v + i\xi \cdot v + \xi v + (\langle v, \xi\rangle + \langle v, i\xi\rangle)(1 + i + \xi + i\xi)) \\ &= \tfrac{1}{2}((-1 + i + \xi + i\xi)v + \langle v, \xi + i\xi\rangle(1 + i + \xi + i\xi))\ , \end{aligned}$$

where in the sixth equality the formulas (4.13), (4.14), (4.15), (4.16) and (4.17) were used, while in seventh equality the rule $v\xi = -\xi v - 2\langle v, \xi\rangle$ was applied. □

Using this result the inverse function $Q_\xi^{-1} : V_\xi \to V_i$ can be calculated as well by observing that the roles of $i$ and $\xi$ are played by $-\xi$ and $-i$ respectively. Taking into account that any $v \in V_\xi$ is perpendicular to $\xi$, virtually the same calculation leads to

$$\begin{aligned} Q_\xi^{-1}(v) &= \bar{r}_\xi v r_\xi = \frac{1}{4}(1 - i - \xi - i\xi)v(1 + i + \xi + i\xi) \\ &= \frac{1}{4}(1 + (-i) + (-\xi) + (-\xi)(-i))v(1 - (-i) - (-\xi) - (-\xi)(-i)) \\ &= (-1 - \xi - i + \xi i)v + (\langle v, -i + \xi i\rangle)(1 - \xi - i + \xi i) \; . \end{aligned}$$

Moreover,

$$Q_{-\xi}^{-1}(v) = (-1 + \xi - i - \xi i)v + (\langle v, -i - \xi i\rangle)(1 + \xi - i - \xi i) \; .$$

**Lemma 4.18.** *If $v, \xi \in V_i$, then*

$$Q_{-\xi}^{-1} \circ Q_\xi(v) = v\xi - \langle v\xi, 1\rangle(1 + \xi) - \langle v\xi, i\rangle(1 + \xi)i \; . \tag{4.18}$$

*Proof.* To calculate $Q_{-\xi} \circ Q_\xi(v)$ for an arbitrary $v \in V_i$ more preparation is needed.

1. Applying (4.1) one has

$$\xi(i\xi \cdot v) + i\xi \cdot \xi v = \xi i\xi \cdot v + i\xi\xi \cdot v = iv - iv = 0 \tag{4.19}$$

2. By changing the order of terms in the multiplications one obtains

$$i \cdot \xi v = v\xi \cdot i - 2\langle \xi v, i\rangle = -vi \cdot \xi - 2\langle \xi v, i\rangle \; ,$$

$$\xi \cdot iv = vi \cdot \xi - 2\langle iv, \xi\rangle \; .$$

   Using (4.5) and the definition of multiplication it can be prooved, that

$$2\langle \xi v, i\rangle - 2\langle iv, \xi\rangle = 4\langle i\xi, v\rangle \; .$$

   Therefore,

$$\begin{aligned} \xi \cdot iv - i \cdot \xi v &= 2vi \cdot \xi + 2\langle \xi v, i\rangle - 2\langle iv, \xi\rangle = 2vi \cdot \xi + 4\langle i\xi, v\rangle \\ &= -2iv \cdot \xi + 4\langle i\xi, v\rangle = 2i\xi \cdot v + 4\langle v, \xi\rangle i + 4\langle i\xi, v\rangle \; , \end{aligned} \tag{4.20}$$

   and thus

$$-2i\xi \cdot v + \xi \cdot iv - i \cdot \xi v = 4\langle v, \xi\rangle i + 4\langle i\xi, v\rangle \; . \tag{4.21}$$

3. By exchanging $\xi$ with $i\xi$ in (4.20) one has

$$i\xi \cdot iv - i(i\xi \cdot v) = 2ii\xi v + 4\langle v, i\xi\rangle i + 4\langle ii\xi, v\rangle = -2\xi v + 4\langle v, i\xi\rangle i - 4\langle \xi, v\rangle \; . \tag{4.22}$$

4. If $a, b \in \mathbb{O}'$ and $a \perp b$, then $ab$ is orthogonal to both $a$ and $b$. Thus

$$\langle 1 + i + \xi + i\xi, -i + i\xi\rangle = 0 - 1 + 0 + 1 = 0 \; . \tag{4.23}$$

5. Finally, taking into account again the orthogonality assumptions and (4.5)
$$\langle i\xi \cdot v, i\rangle = -\langle \xi i \cdot v, i\rangle = \langle iv, \xi i\rangle + 2\underbrace{\langle i, \xi i\rangle\langle v, 1\rangle}_{0} = -\langle iv, i\xi\rangle = -\langle v, \xi\rangle\ .$$
As a consequence, this leads to
$$\begin{gathered}\langle(-1+i+\xi+i\xi)v, -i+i\xi\rangle = \\ \underbrace{\langle -v, -i+i\xi\rangle}_{\langle -v, i\xi\rangle} + \underbrace{\langle iv, -i+i\xi\rangle}_{\langle iv, i\xi\rangle} + \langle \xi v, -i+i\xi\rangle + \underbrace{\langle i\xi\cdot v, -i+i\xi\rangle}_{\langle i\xi\cdot v, -i\rangle} = \\ \langle -v, i\xi\rangle + \underbrace{\langle iv, i\xi\rangle}_{\langle v,\xi\rangle} - \underbrace{\langle \xi v, -i\rangle}_{-\langle v, \xi i\rangle} - \underbrace{\langle \xi v, \xi i\rangle}_{\langle v, i\rangle = 0} - (-\langle v, \xi\rangle) = 2\langle v, \xi i\rangle + 2\langle v, \xi\rangle\ . \end{gathered} \tag{4.24}$$

To simplify calculation it is useful to get rid of the constant factor. According to Lemma 4.17 we have
$$\begin{aligned}
4Q_{-\xi}^{-1}\circ Q_\xi(v) &= (-1-i+\xi+i\xi)\left((-1+i+\xi+i\xi)v + \langle v, \xi+i\xi\rangle(1+i+\xi+i\xi)\right) \\
&\quad + \langle(-1+i+\xi+i\xi)v + (\langle v, \xi+i\xi\rangle)(1+i+\xi+i\xi), -i-\xi i\rangle\cdot \\
&\quad \cdot(1-i+\xi+i\xi) \\
&= v - iv - \xi v - i\xi\cdot v + \langle v, \xi+i\xi\rangle(-1-i-\xi-i\xi) \\
&\quad + iv - i^2 v - i\cdot\xi v - i(i\xi\cdot v) + \langle v, \xi+i\xi\rangle(-i-i^2-i\xi-i^2\xi) \\
&\quad -\xi v + \xi\cdot iv + \xi^2 v + \xi(i\xi\cdot v) + \langle v, \xi+i\xi\rangle(\xi+\xi i+\xi^2+\xi i\xi) \\
&\quad -i\xi\cdot v + i\xi\cdot iv + i\xi\cdot\xi v + (i\xi)^2 v + \langle v, \xi+i\xi\rangle(i\xi+i\xi i+i\xi^2+(i\xi)^2) \\
&\quad + [\langle(-1+i+\xi+i\xi)v, -i+i\xi\rangle \\
&\quad + \langle v, \xi+i\xi\rangle\langle 1+i+\xi+i\xi, -i+i\xi\rangle]\,(1-i+\xi+i\xi) \\
&= -2\xi v + (-2i\xi\cdot v + \xi\cdot iv - i\cdot\xi v) \\
&\quad + (i\xi\cdot iv - i(i\xi\cdot v)) + (\xi(i\xi\cdot v) + i\xi\cdot\xi v) \\
&\quad + 2\langle v, \xi+i\xi\rangle(-1-i+\xi+\xi i) \\
&\quad + \langle(-1+i+\xi+i\xi)v, -i+i\xi\rangle(1+\xi-i+i\xi) \\
&= -4\xi v - 4\langle\xi, v\rangle + 4\langle v, \xi\rangle i + 4\langle i\xi, v\rangle + 4\langle v, i\xi\rangle i + \\
&\quad 2(\langle v, \xi+i\xi\rangle)(-1-i+\xi+\xi i) + 2(\langle v, \xi+i\xi\rangle)(1+\xi-i+i\xi) \\
&= -4\xi v + \langle\xi, v\rangle(-4-2+2+4i-2i-2i+2\xi+2\xi+2\xi i+2i\xi) + \\
&\quad \langle i\xi, v\rangle(4-2-2+4i-2i+2i+2\xi-2\xi+2\xi i-2i\xi) \\
&= -4\xi v + \langle\xi, v\rangle(-4+4\xi) + \langle i\xi, v\rangle(4i+4\xi i) \\
&= 4v\xi + \langle\xi, v\rangle(4+4\xi) + \langle i\xi, v\rangle(4i+4\xi i)\ ,
\end{aligned}$$
where in the fourth equality the formulas (4.19), (4.21), (4.22), (4.23) and (4.24) were used. To sum it up, the required transformation is given by
$$\begin{aligned} Q_{-\xi}^{-1}\circ Q_\xi(v) &= v\xi + \langle\xi, v\rangle(1+\xi) + \langle i\xi, v\rangle(1+\xi)i \\ &= v\xi - \langle v\xi, 1\rangle(1+\xi) - \langle v\xi, i\rangle(1+\xi)i\ . \end{aligned}$$
□

Now we substitute $j, e$ and $g$ into $v$. As mentioned earlier, the subspace $V_i$ is a complex linear space with basis $j, e, g$ and complex structure $J_i : V_i \to V_i$, $v \mapsto iv$. Since $\xi \in V_i$, the coordinate expression of $\xi$ in $V_i$ can be written as
$$\begin{aligned} \xi = uj + ve + wg &= (u_1+u_2 I)j + (v_1+v_2 I)e + (w_1+w_2 I)g \\ &= u_1 j + u_2 k + v_1 e + v_2 f + w_1 g - w_2 h\ , \end{aligned}$$

where $u, v, w \in \mathbb{C}$, $u_i, v_i, w_i \in \mathbb{R}$ for $i = 1, 2$. Because $V_i = V_{-i}$ as a subspace, $\xi$ can be expressed as a element of $V_{-i}$ as well. Here the basis is the same, but the complex structure is given by $J_{-i} : V_{-i} \to V_{-i}$, $v \mapsto -iv$. Therefore, the coordinate expression of the same $\xi$ here is

$$\xi = \overline{u}j + \overline{v}e + \overline{w}g = (u_1 - u_2 I)j + (v_1 - v_2 I)e + (w_1 - w_2 I)g \ .$$

According to the multiplication rule of the basis vectors of $\mathbb{O}$ (which is represented by the Fano-plane) it is possible to compute the multiplication of $\xi$ with the basis vectors from the left:

$$j\xi = -u_1 + u_2 i + v_1 g + v_2 h - w_1 e + w_2 f = -u \cdot 1 + 0j - we + vg \ , \tag{4.25}$$

$$e\xi = -u_1 g - u_2 h - v_1 + v_2 i + w_1 j - w_2 k = -v \cdot 1 + wj + 0e - ug \ , \tag{4.26}$$

$$g\xi = u_1 e - u_2 f - v_1 j + v_2 k - w_1 + w_2 i = -w \cdot 1 - vj + ue + 0g \ , \tag{4.27}$$

because the resulting vector $v$, of which the terms are calculated here, is in $V_{-i}$. Similarly,

$$\begin{aligned} \xi i &= -u_1 k + u_2 j - v_1 f + v_2 e + w_1 h + w_2 g \\ &= (u_2 + u_1 I)j + (v_2 + v_1 I)e + (w_2 + w_1 I)g \\ &= (\overline{u}I)j + (\overline{v}I)e + (\overline{w}I)g \ . \end{aligned} \tag{4.28}$$

Then,

$$Q_{-\xi}^{-1} \circ Q_\xi(j) = j\xi - \langle j\xi, 1\rangle(1+\xi) - \langle j\xi, i\rangle(1+\xi)i =$$

$$\begin{pmatrix} 0 \\ -w \\ v \end{pmatrix} + u_1 \begin{pmatrix} \overline{u} \\ \overline{v} \\ \overline{w} \end{pmatrix} - u_2 \begin{pmatrix} \overline{u}I \\ \overline{v}I \\ \overline{w}I \end{pmatrix} = \begin{pmatrix} 0 \\ -w \\ v \end{pmatrix} + \underbrace{(u_1 - u_2 I)}_{\overline{u}} \begin{pmatrix} \overline{u} \\ \overline{v} \\ \overline{w} \end{pmatrix} = \begin{pmatrix} \overline{u}^2 \\ \overline{uv} - w \\ \overline{uw} + v \end{pmatrix} ,$$

$$Q_{-\xi}^{-1} \circ Q_\xi(e) = e\xi - \langle e\xi, 1\rangle(1+\xi) - \langle e\xi, i\rangle(1+\xi)i =$$

$$\begin{pmatrix} w \\ 0 \\ -u \end{pmatrix} + v_1 \begin{pmatrix} \overline{u} \\ \overline{v} \\ \overline{w} \end{pmatrix} - v_2 \begin{pmatrix} \overline{u}I \\ \overline{v}I \\ \overline{w}I \end{pmatrix} = \begin{pmatrix} w \\ 0 \\ -u \end{pmatrix} + \underbrace{(v_1 - v_2 I)}_{\overline{v}} \begin{pmatrix} \overline{u} \\ \overline{v} \\ \overline{w} \end{pmatrix} = \begin{pmatrix} \overline{vu} + w \\ \overline{v}^2 \\ \overline{vw} - u \end{pmatrix} ,$$

$$Q_{-\xi}^{-1} \circ Q_\xi(g) = g\xi - \langle g\xi, 1\rangle(1+\xi) - \langle g\xi, i\rangle(1+\xi)i =$$

$$\begin{pmatrix} -v \\ u \\ 0 \end{pmatrix} + w_1 \begin{pmatrix} \overline{u} \\ \overline{v} \\ \overline{w} \end{pmatrix} - w_2 \begin{pmatrix} \overline{u}I \\ \overline{v}I \\ \overline{w}I \end{pmatrix} = \begin{pmatrix} -v \\ u \\ 0 \end{pmatrix} + \underbrace{(w_1 - w_2 I)}_{\overline{w}} \begin{pmatrix} \overline{u} \\ \overline{v} \\ \overline{w} \end{pmatrix} = \begin{pmatrix} \overline{wu} - v \\ \overline{wv} + u \\ \overline{w}^2 \end{pmatrix} .$$

Putting all together, the matrix which represents the mapping $Q_{-\xi}^{-1} \circ Q_\xi : V_i \to V_{-i}$ is

$$M_{\overline{\xi}} = \begin{pmatrix} \overline{u}^2 & \overline{vu} + w & \overline{wu} - v \\ \overline{uv} - w & \overline{v}^2 & \overline{wv} + u \\ \overline{uw} + v & \overline{vw} - u & \overline{w}^2 \end{pmatrix} ,$$

and to get matrix of the same function as a $V_{-i} \to V_{-i}$ mapping each complex coordinate of $\xi$ should be conjugated:

$$M_\xi = \begin{pmatrix} u^2 & vu + \overline{w} & wu - \overline{v} \\ uv - \overline{w} & v^2 & wv + \overline{u} \\ uw + \overline{v} & vw - \overline{u} & w^2 \end{pmatrix} .$$

**Corollary 4.19.** *The transition function between the two trivializations of the principal $SU(3)$-bundle $G_2 \to S^6$ at the equator is*

$$\theta\colon S^5 \to SU(3), \quad \theta\begin{pmatrix} u \\ v \\ w \end{pmatrix} \mapsto \begin{pmatrix} u^2 & vu+\overline{w} & wu-\overline{v} \\ uv-\overline{w} & v^2 & wv+\overline{u} \\ uw+\overline{v} & vw-\overline{u} & w^2 \end{pmatrix} .$$

According to Corollary 2.51 the principal $SU(3)$-bundles over $S^6$ are classified by $\pi_5(SU(3))$.

**Theorem 4.20.** $\pi_5(SU(3)) = \mathbb{Z}$.

*Proof (Sketch).* From the well-known periodicity theorem of Bott [4] it follows that $\pi_5(SU(4)) = \mathbb{Z}$. It can be shown as well that $SU(4) = Spin(6)$. By definition $Spin(6)$ is the double cover of $SO(6)$. A covering mapping induces isomorphisms on the higher homotopy groups of the total and base spaces. Thus, $\pi_5(Spin(6)) = \pi_5(SO(6))$. Moreover, $\mathbb{C}P^3 = SO(6)/U(3)$ and from the long exact sequence of this fibration one obtains $\pi_5(SO(6)) = \pi_5(U(3))$. Finally the mapping $\det : U(3) \to U(1)$ is a locally trivial fibration with fibers $\det^{-1}(1) = SU(3)$ and from the long exact sequence of this fibration one obtains $\pi_5(U(3)) = \pi_5(SU(3))$. □

**Proposition 4.21** ([7])**.** $\theta\colon S^5 \to SU(3)$ *is the generator of* $\pi_5(SU(3))$.

*Proof.* The columns of a matrix in $SU(3)$ are unit length vectors in $\mathbb{C}^3$. Define a mapping $\pi : SU(3) \to S^5$ as the projection onto the first column. Then the fiber above e.g. $(1,0,0)$ is $SU(2)$ and therefore $\pi : SU(3) \to S^5$ is a fibration with fibers $SU(2)$. Then the long exact homotopy sequence of this fibrations gives

$$\underbrace{\pi_5(SU(3))}_{\mathbb{Z}} \xrightarrow{\pi_*} \underbrace{\pi_5(S^5)}_{\mathbb{Z}} \longrightarrow \underbrace{\pi_4(SU(2))}_{\mathbb{Z}_2} \longrightarrow \underbrace{\pi_4(SU(3))}_{0} .$$

Because the mapping $\pi_4(SU(2)) \to \pi_4(SU(3))$ is surjective, the map $\pi_*$ should be a multiplication with 2. A generator of $\pi_5(S^5)$ is just a map $S^5 \to S^5$ of degree one. The degree of $\pi \circ \theta\colon S^5 \to S^5$ is 2, because this mapping is just the first column of $\theta$. For example, the point $(1,0,0)$ has preimage $\{(1,0,0),(-1,0,0)\}$. It can be checked that the corresponding signes are the same and therefore $\pi_*([\theta]) = 2$. Thus $[\theta]$ is a generator of $\pi_5(SU(3))$. □

That is, calculating the transition function of the trivializations of the fibration $G_2 \xrightarrow{SU(3)} S^6$ gives a new method for calculating the generator of $\pi_5(SU(3))$. The resulting function coincides with the functions given in [7] and [18].

### 4.2.2 Recovering the group structure

According to Corollary 4.19 the space $G_2$ can be constructed as the factor space

$$(D^6_+ \times SU(3) \cup D^6_- \times SU(3))/\sim ,$$

where $(\xi_1, x) \sim (\xi_2, y)$ if and only if $(\xi_1, x), (\xi_2, y) \in S^5 \times SU(3)$, $\xi_1 = \xi_2$, and $y = \theta(\xi_1)x$. It is equivalent to construct it as

$$((S^6 \setminus \{S\}) \times SU(3) \cup (S^6 \setminus \{N\}) \times SU(3))/\sim ,$$

where $(\xi_1, x) \sim (\xi_2, y)$ if and only if $(\xi_1, x), (\xi_2, y) \in (S^6 \setminus \{S, N\}) \times SU(3)$, $\xi_1 = \xi_2$, $y = \tau_{12}(\xi_1)x$, and $\tau_{12}$ is the transition function on $S^6 \setminus \{S, N\}$.

On this space it is possible to recover the group structure of $G_2$. Consider $(S^6 \setminus \{S\}) \times SU(3)$. Due to the construction of trivialization, the matrices in the fiber of $\xi$ are operators of $V_\xi$ written in the basis $\{Q_\xi(j), Q_\xi(e), Q_\xi(g)\}$. This is equivalent to say that $\xi$ and $V_\xi$ were rotated to $i$ and $V_1$ respectively with $Q_\xi^{-1}$. Then the matrix of the rotated operator in the basis $\{j, e, g\}$ is the same as the original operator in the basis $\{Q_\xi(j), Q_\xi(e), Q_\xi(g)\}$. This follows from the fact that $Q_\xi$ is an orthogonal operator and then e.g. the real component of the first element of the column is

$$\langle \eta, Q_\xi(j) \rangle = \langle Q_\xi^T(\eta), j \rangle = \langle Q_\xi^{-1}(\eta), j \rangle \ .$$

In other words, for a general $\varphi \in G_2$ which is represented by $(\xi, \eta, \zeta)$ the trivializaiton is

$$\begin{aligned}\psi_1(\varphi) = \psi_1(\xi, \eta, \zeta) &= (\xi, [\varphi(j)|\varphi(e)|\varphi(g)]_{\{Q_\xi(j), Q_\xi(e), Q_\xi(g)\}}) \\ &= (\xi, [Q_\xi^{-1} \cdot \varphi(j)|Q_\xi^{-1} \cdot \varphi(e)|Q_\xi^{-1} \cdot \varphi(g)]_{\{j,e,g\}}) \ .\end{aligned}$$

which means, that the octonion component describes the image of $i$ in $S^6$ and the matrix component describes the image of $\{j, e, g\}$ in the tangent space of corresponding point of $S^6$ relatively to the translated basis vectors. As a consequence, the octonion and matrix components behave independently and to get the matrix component of two consecutive automorphisms the matrices should just be multiplied together in the regular way.

The octonion component should record the image of $i$. In the cases above this is given by $\xi_1 = Q_{\xi_1}(i)$ and $\xi_2 = Q_{\xi_2}(i)$. This component of the composed automorphism can be calculated by composing the corresponding $Q_\xi$ functions. That is, the image of $i$ will be $Q_{\xi_1} \circ Q_{\xi_2}(i)$. Putting all together, the multiplication is defined as

$$(\xi_1, M_1) \cdot (\xi_2, M_2) = (Q_{\xi_1} \circ Q_{\xi_2}(i), M_1 M_2) \ .$$

If one of the terms, e.g. $(\xi_1, M_1)$ is given with the trivialization $\psi_2$, then the matrix component should be first transformed by $\tau_{12}(\xi_1)$ to get the same element in the other trivialization. This is given by $\theta(\xi_1)$ if $\xi$ is in the equator, but generally more calculation is needed.

## 4.3 $G_2$ as the stabilizer of a 3-form

In the preceeding sections we defined $G_2$ as the automorphism group of the Cayley algebra. It is a very interesing fact that there is seemingly different definition (or construction) of the group $G_2$, which is indeed related to the octonions [5, 13]. As a concluding remark we briefly summarize this construction.

Suppose that $V$ is a vector space and $f : V^* \to V^*$ is a linear transformation of the dual space. The graded exterior algebra of $V^*$ will be denoted by $\Lambda(V^*)$. By the universal property of the exterior product of vector spaces, there exists a unique homomorphism of graded algebras $\Lambda(f) : \Lambda(V^*) \to \Lambda(V^*)$ such that

$$\Lambda(f)|_{\Lambda^1(V^*)} = f \ .$$

In particular, $\Lambda(f)$ preserves homogeneous degree. The $k$-graded components of $\Lambda(f)$ are given by

$$\Lambda(f)(x^1 \wedge \cdots \wedge x^k) = f(x^1) \wedge \cdots \wedge f(x^k) \ .$$

From now on let $V = \mathbb{R}^7$ and let $(e_1, \ldots, e_7)$ be a basis of $V$. Write $e^{i_1 \cdots i_l}$ for the exterior form $e^{i_1} \wedge \cdots \wedge e^{i_l}$ where $(e^1, \ldots, e^7)$ is the dual basis. Define a 3-form $\varphi$ on $\mathbb{R}^7$ by

$$\varphi = e^{123} + e^{145} - e^{167} + e^{246} + e^{257} + e^{347} - e^{356} \; . \tag{4.29}$$

A linear map $f : V^* \to V^*$ *preserves* $\varphi$ if

$$\Lambda(f)(\varphi) = \varphi \; .$$

**Theorem 4.22.** *The subgroup of $GL(V^*) = GL(7, \mathbb{R})$ preserving $\varphi$ is the Lie group $G_2$.*

*Proof.* For the proof of the theorem we define a nondegenerate bilinear symmetric form. For each $x \in V$ it is possible to define an antiderivation of the algebra $\Lambda(V^*)$ called *interior product* (or *contraction*)

$$i_x \colon \Lambda^k(V^*) \to \Lambda^{k-1}(V^*), \quad \omega(u^1, \ldots, u^k) \mapsto \omega(x, u^2, \ldots, u^k);.$$

Now the bilinear form induced by $\varphi$ (see [5],[8]) is defined as

$$\beta_\varphi \colon V \times V \to \Lambda^7(V^*) \simeq \mathbb{R}, \quad (x, y) \mapsto -\frac{1}{6} i_x\varphi \wedge i_y\varphi \wedge \varphi \; .$$

Since two forms commute this is symmetric in $x, y \in V = \mathbb{R}^7$ and it can be shown that this form is nondegenerate. The isotropy (stabilizer) group of this special 7 dimensional metric is in $SO(7)$.

An algebra in general is given by a particular $T^2_1$ tensor: the product $xy = z$ means precisely this. Since $\beta_\varphi$ is nondegenerate, it defines an isomorphism $B_\varphi : V \to V^*$ as follows. For each $x$ in $V$ we let $x^* \in V^*$ be

$$x^* \colon V \mapsto \mathbb{R}, \quad y \mapsto \beta_\varphi(x, y) \; .$$

Using this isomorphism we flip an index of $\varphi \in \Lambda^3_0(V) \subset T^3_0(V)$ to get a tensor in $T^2_1(V)$, denoted by $\gamma$. In particular

$$\gamma_\varphi(x, y) = B^{-1}_\varphi(i_y i_x \varphi) \; .$$

Hence, in $V$ one gets an algebra with $\gamma(x, y) = z$. This form is antisymmetric for the basis $e_1, \ldots, e_7$:

$$\gamma_\varphi(e_1, e_2) = e_1 e_2 = -e_2 e_1$$

because so is $\varphi$ in two indices. Moreover it is alternative because $\varphi$ is fully antisymmetric. Because of the choice of $\varphi$ this reproduces the product of imaginary octonions with the following identification of the basis vectors:

$$e_1 = i, \; e_2 = j, \; e_3 = k, \; e_4 = e, \; e_5 = f, \; e_6 = g, \; e_7 = g \; .$$

This can be checked with easy calculation.

As an example the product of $i$ and $j$,

$$\gamma_\varphi(e_1, e_2) = B^{-1}_\varphi(\varphi(e_1, e_2, \cdot)) = B^{-1}_\varphi(e^1 \wedge e^2 \wedge e^3(e_1, e_2, \cdot)) = B^{-1}_\varphi(e^3)$$

will be computed. To get the result of the multiplicaiton it is necessary to find $B^{-1}_\varphi(e^3)$, i.e. to find the specific vector $x \in V$ such that for all $y \in V$

$$\beta_\varphi(x, y) = e^3(y) \; .$$

For example

$$\begin{aligned}\beta_\varphi(e_3,e_3) &= -\tfrac{1}{6} i_{e_3}\varphi\wedge i_{e_3}\varphi\wedge\varphi\\ &= -\tfrac{1}{6}(e^{12}+e^{47}-e^{56})\wedge(e^{12}+e^{47}-e^{56})\wedge\varphi\\ &= -\tfrac{1}{6}(-2e^{12}\wedge e^{47}\wedge e^{356}-2e^{12}\wedge e^{56}\wedge e^{347}-2e^{47}\wedge e^{56}\wedge e^{123})\\ &= -\tfrac{1}{6}(-6)=1\ ,\end{aligned}$$

where we used the identification $e^{1234567}=1$. Moreover,

$$\begin{aligned}\beta_\varphi(e_3,e_1) &= -\tfrac{1}{6} i_{e_3}\varphi\wedge i_{e_1}\varphi\wedge\varphi\\ &= -\tfrac{1}{6}(e^{12}+e^{47}-e^{56})\wedge(e^{23}+e^{45}-e^{67})\wedge\varphi\\ &= -\tfrac{1}{6}(e^{12}\wedge e^{45}\wedge 0-e^{12}\wedge e^{67}\wedge 0+e^{47}\wedge e^{23}\wedge 0-e^{56}\wedge e^{23}\wedge 0)=0\ .\end{aligned}$$

With similar calculations one gets that $\beta_\varphi(e_3,e_i)=\delta_{3i}$, and thus $\beta_\varphi(e_3,y)=e^3(y)$. This means that $\gamma_\varphi(e_1,e_2)=e_3$, i.e. $ij=k$. By adding the unit 1, and redefining $e_i^2$ to be $-1$ intead of 0 the octonion composition and division algebra is reconstructed.

Because of the construction a map $f\in SO(7)$ preserves $\varphi$ if and only if it preserves the multiplication tensor $\gamma_\varphi$. Therefore the group $G_2$ is the isotropy group of $\varphi$. □

### 4.3.1 $G_2$-manifolds

Let $M$ be a manifold, let $g$ be a Riemannian metric on $M$ and let $\nabla$ be the Levi-Civita connection of $g$ (for a detailed treatment of Riemannian geometry the reader is referred to [3, 13]). Let $x,y$ be points in $M$ joined by a smooth path $\gamma$. Then the parallel transport along $\gamma$ using $\nabla$ defines an isometry between the tangent spaces $T_xM$ and $T_yM$.

**Definition 4.23** ([12]). The *holonomy group* $\mathrm{Hol}(g)$ of $g$ is the group of isometries of $T_xM$ generated by parallel transport around closed loops based at $x\in M$. $\mathrm{Hol}(g)$ can be considered as a subgroup of $O(n)$, defined up to conjugation by elements of $O(n)$. Then $\mathrm{Hol}(g)$ is independent of the base point $x$.

**Theorem 4.24** ([2]). *Let $M$ be a simply-connected, $n$ dimensional Riemannian manifold with metric $g$ which is irreducible (not locally a product space) and nonsymmetric (not locally a Riemannian symmetric space). Then either*

*(i)* $\mathrm{Hol}(g)=SO(n)$

*(ii)* $\mathrm{Hol}(g)=SU(m)$ *or* $U(m)$ *and* $n=2m$

*(iii)* $\mathrm{Hol}(g)=Sp(m)$ *or* $Sp(m)Sp(1)$ *and* $n=4m$

*(iv)* $\mathrm{Hol}(g)=G_2(m)$ *and* $n=7$

*(v)* $\mathrm{Hol}(g)=Spin(7)$ *and* $n=8$.

**Definition 4.25.** The *frame bundle* $F$ of $M$ is the bundle over $M$ whose fibre at $p\in M$ is the set of isomorphism between $T_pM$ and $\mathbb{R}^7$. It can be shown that $F$ is a principal $GL(n,\mathbb{R})$-bundle over $M$. Let $G$ be a Lie subgroup of $GL(n,\mathbb{R})$. Then a *G-structure* on $M$ is a principal subbundle $P$ of $F$, with fibre $G$.

Let $M$ be an oriented 7-manifold. For each $p\in M$ define $P^3_pM$ to be the subset of 3-form $\varphi\in\Lambda^3T^*_pM$ for which there exists an oriented isomorphism between $T_pM$ and $\mathbb{R}^7$ identifying $\omega$ and the 3-form defined in (4.29). Then $P^3_pM$ is isomorphic to $GL_+(7,\mathbb{R})/G_2$ since $\varphi$ has symmetry group $G_2$. By dimension arguments the bundle $P^3M=\coprod_p P^3_pM$

is an open subbundle of $\Lambda^3 T^*M$ with fibre $GL_+(7,\mathbb{R})/G_2$. A 3-form $\omega$ on $M$ is said to be *positive* if $\omega|_p \in P^3_p M$ for all $p \in M$.

Let $\omega$ be a positive form on $M$, and let $Q$ be the subset of the frame bundle $F$ consisting of isomorphism between $T_pM$ and $\mathbb{R}^7$ which identify $\omega|_p$ and $\varphi$ of (4.29). It can be shown that $Q$ is a principal subbundle of $F$, with fibre $G_2$. That is, $Q$ is a $G_2$-structure. Conversely, every $G_2$-structure defines a 3-form $\omega$ and a metric $g$ on $M$ corresponding to $\varphi$ and the standard Euclidean metric on $\mathbb{R}^7$. Then the associated pair $(\omega, g)$ is also called a $G_2$-structure.

**Definition 4.26.** Let $M$ be a 7-dimensional manifold, $(\omega, g)$ a $G_2$-structure on $M$, and $\nabla$ the Levi-Civita conenction of $g$. Then $\nabla\varphi$ is the torsion of $(\omega, g)$, and $(\omega, g)$ is *torsion-free* if $\nabla\varphi = 0$. A *$G_2$-manifold* is a triple $(M, \varphi, g)$ where $(\varphi, g)$ is a torsion-free $G_2$-structure on $M$. It can be shown that for the induced metric $\mathrm{Hol}(g) \subseteq G_2$.

**Theorem 4.27.** *Let $(M, \varphi, g)$ a compact $G_2$-manifold. Then $\mathrm{Hol}(g) = G_2$ if and only if $\pi_1(M)$ is finite.*

The first complete, but noncompact 7-manifolds with holonomy $G_2$ were constructed by Robert Bryant and Salamon in 1989 [6]. The first compact 7-manifolds with holonomy $G_2$ were constructed by Dominic Joyce in 1994 [11], and compact $G_2$ manifolds are sometimes known as "Joyce manifolds", especially in the physics literature.

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