Solutions to January 2011 Problems

Problem 1. (a) Let n be even, and let $\mathcal{P}$ be a regular n-sided polygon which is inscribed in a circle of radius 1. Show that for any point A on that circle, the sum of the squares of the distances from A to the vertices of $\mathcal{P}$ is $2n$. (b) What about if n is odd? (This is probably quite a bit harder.)

Solution. (a) Divide the n vertices of $\mathcal{P}$ into $n/2$ diametrically opposite pairs. If $\{U, V\}$ is such a pair, then since UV is a diameter, $\triangle UAV$ is right-angled at A. Thus, by the Pythagorean Theorem, $(AU)^2 + (AV)^2 = (UV)^2 = 4$. (In the special case where A is one of U or V, we don't quite have a triangle, but even more easily we have $(AU)^2 + (AV)^2 = 4$.) Now sum over all $n/2$ pairs. Our sum is $(n/2)(4)$, that is, $2n$.

(b) We can take the n vertices of the circle to be $(\cos\theta_k, \sin\theta_k)$ where $\theta_k = 2\pi k/n)$ (k= 0, 1, 2, …, $n-1$.) Let A have coordinates $(\cos\alpha, \sin\alpha)$. Then the distance from $(\cos\alpha, \sin\alpha)$ to $(\cos\theta_k, \sin\theta_k)$ is equal to

$$\sqrt{(\cos\alpha - \cos\theta_k)^2 + (\sin\alpha - \sin\theta_k)^2}.$$

Thus the sum we are looking for is

$$\sum_{k=0}^{n-1}((\cos\alpha - \cos\theta_k)^2 + (\sin\alpha - \sin\theta_k)^2).$$

Expand the squares and simplify, using the fact that $\cos^2 x + \sin^2 x = 1$. After a while we obtain

$$2n - 2\sum_{k=0}^{n-1}(\cos\alpha\cos\theta_k + \sin\alpha\sin\theta_k).$$

We complete the calculation by showing that $\sum\cos\theta_k$ and $\sum\sin\theta_k$ are both 0. The easiest way to do this is to use the fact that $\cos\phi + i\sin\phi = e^{i\phi}$. But we can also prove the results by using trigonometric identities. The sum involving sines is very easy. All we need is the fact that $\sin(2\pi - \phi) = -\sin\phi$. The sum involving cosines is more difficult. Let $\beta = \pi/n$. We want to calculate

$$\cos 0 + \cos 2\beta + \cos 4\beta + \cdots + \cos 2(n-1)\beta.$$

Multiply each term by $\sin\beta$. Now note that

$$2\sin\beta\cos 2k\beta = \sin(2k+1)\beta - \sin(2k-1)\beta.$$

(This fact is readily checked by using the ordinary formulas for the sine of a sum and difference of two numbers). Now we can use the above fact to simplify

$$2\sin\beta\cos 0 + 2\sin\beta\cos 2\beta + 2\sin\beta\cos 4\beta + \cdots + 2\sin\beta\cos 2(n-1)\beta.$$

We get

$$[\sin\beta - \sin(-\beta)] + [\sin 3\beta - \sin\beta] + [\sin 5\beta - \sin 3\beta] + \cdots + [\sin(2n-1)\beta - \sin(2n-3)\beta)].$$

Add up, noting the cancellations. There is almost complete collapse. We end up with $-\sin(-\beta) + \sin(2n-1)\beta$, that is, $-\sin(-\pi/n) + \sin(2\pi - \pi/n)$, which is easily seen to be 0.

Comment. A *much* tidier-looking version of the same argument uses some basic vector manipulations. Put the origin O at the centre of the circle. Let the vertices of $\mathcal{P}$ be $V_1, V_2, \ldots, V_n$. We can think of A and the V_i as vectors ("arrows") from O to A and the V_i.

The distance from A to V_i is the length of the vector $V_i - A$. In symbols, it is $\|V_i - A\|$. So we want

$$\|V_1 - A\|^2 + \|V_2 - A\|^2 + \cdots + \|V_n - A\|^2.$$

But in general $\|X - Y\|^2 = \|X\|^2 + \|Y\|^2 - 2X \cdot Y$. So our sum of squares of distances is

$$\|V_1\|^2 + \|V_2\|^2 + \cdots + \|V_n\|^2 + n\|A\|^2 - 2A \cdot (V_1 + V_2 + \cdots + V_n).$$

Each of the $\|V_i\|^2$ is equal to 1, the square of the radius of the circle, as is $\|A\|^2$. By symmetry, the sum of the V_i is the 0-vector. Thus the above sum is equal to $2n$.

Problem 2. Suppose that a and b are positive integers, and that n is an integer greater than 1. Show that if $a^n + b^n$ is a power of 2, then $a = b = 2^e$ for some non-negative integer e.

Solution. Let 2^e be the largest power of 2 that divides *both* a and b. Then $a = 2^e c$, $b = 2^e d$, and at least one of c and d is odd. We will show that in fact $c = d = 1$, which completes the proof.

Since $a^n + b^n$ is a power of 2, $2^{en}c^n + 2^{en}d^n$ is a power of 2. It follows that $c^n + d^n$ is a power of 2. Since c and d are positive, $c^n + d^n$ is even. But then since at least one of c and d is odd, both are odd.

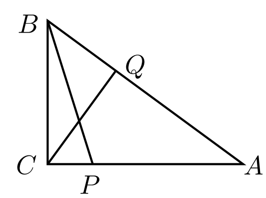
We show that n cannot be even. If n is even, then c^n is a perfect square, and thus has remainder 1 on division by 4. The same remark applies to d^n. So $c^n + d^n$ has remainder 2 on division by 4. But $c^n + d^n$ is a power of 2 that is greater than 2, so it has remainder 0 on division by 4.

So n is odd. Thus

$$c^n + d^n = (c + d)(c^{n-1} - c^{n-2}d + \cdots + d^{n-1}).$$

Note that $c^{n-1} - c^{n-2}d + \cdots + d^{n-1}$ is a sum of an odd number of odd terms. If at least one of c and d is greater than 1, then $c^n + d^n > c + d$. It follows that $c^{n-1} - c^{n-2}d + \cdots + d^{n-1}$ is an odd number greater than 1, so $c^n + d^n$ cannot be a power of 2. We conclude that $c = d = 1$.

Problem 3. Triangle ABC is right-angled at C, CQ is an altitude of $\triangle ABC$, and $AQ = 1$. Point P on CA has the property that $AP = BP = 1$. Find an exact expression for the length of AB.



Solution. There are many approaches. Probably the one that involves the least thinking is to let $t = CP$ and use the Pythagorean Theorem to compute everything in sight. Then some pair of similar triangles and some careful algebra give us t, and then AB.

For fun we will avoid the Pythagorean Theorem, and most of the algebra. Let $z = AB$. Triangles ABC and ACQ can be viewed as having the same height CQ, and bases z and 1. Thus the ratio of their areas is $z/1$. But the triangles are similar, with corresponding sides AC and AQ, so the ratio of their areas is $(AC)^2/1^2$. It follows that $(AC)^2 = z$, so $AC = \sqrt{z}$.

Now drop a perpendicular from P to AB, meeting AB at R (not shown). Then $\triangle APR$ is similar to $\triangle ABC$. It follows that $AR/AP = AC/AB$. Note that $AR = z/2$. So $(z/2)/1 = \sqrt{z}/z$. This can be rewritten as $z^{3/2} = 2$, so $z = 2^{2/3}$.

Problem 4. A fair die has the numbers 0, 0, 0, 0, 0, and 1 written on its six faces. This die is tossed n times. Find a simple expression for the probability that the sum of the n numbers obtained is odd. The expression should not involve the summation operator $\sum$.

Solution. Let p_n be the required probability. We find an expression for p_{n+1} in terms of p_n. If the die is tossed $n+1$ times, the sum of the $n+1$ numbers obtained can be odd in two different ways: (1) The sum of the first n numbers is odd, and the $(n+1)$-th toss is a 0 or (2) The sum of the first n numbers is even, and the $(n+1)$-th toss is a 1.

The probability of (1) is $5p_n/6$, and the probability of (2) is $(1-p_n)/6$. It follows that

$$p_{n+1} = \frac{5p_n}{6} + \frac{1-p_n}{6} = \frac{1}{6} + \frac{2p_n}{3}.$$

Note that $p_1 = 1/6$. Thus $p_2 = 1/6 + (1/6)(2/3)$, and $p_3 = 1/6 + (1/6)(2/3) + (1/6)(2/3)^2$, and in general

$$p_n = \frac{1}{6} + \frac{1}{6}\left(\frac{2}{3}\right) + \cdots + \frac{1}{6}\left(\frac{2}{3}\right)^{n-1}.$$

The above geometric progression can be evaluated in the usual way. We obtain

$$p_n = \frac{1}{2} - \frac{1}{2}\left(\frac{2}{3}\right)^n.$$

Another Way. We solved the recurrence using a *general* strategy, one that will work with $x_{n+1} = a + bx_n$, where $b \neq 0$. In our case, we can take a shortcut. It is intuitively clear that p_n is close to $1/2$ when n is large. So let $p_n = 1/2 - x_n$. Then

$$\frac{1}{2} - x_{n+1} = \frac{1}{6} + \frac{2}{3}\left(\frac{1}{2} - x_n\right).$$

This simplifies to $x_{n+1} = (2/3)x_n$. But it is easy to see that $x_1 = 1/3$, so $x_n = (1/3)(2/3)^{n-1}$.

Another Way. The sum is odd if and only if there is an odd number of 1's. The probability that there are k 1's is $\binom{n}{k}(1/6)^k(5/6)^{n-k}$. We want to find the sum of these probabilities over all odd k. So we want to find a simple expression for

$$\binom{n}{1}(5/6)^{n-1}(1/6)^1 + \binom{n}{3}(5/6)^{n-3}(1/6)^3 + \binom{n}{5}(5/6)^{n-5}(1/6)^5 \cdots.$$

By the Binomial Theorem,

$$1 = (5/6+1/6)^n = \binom{n}{0}(5/6)^n(1/6)^0 + \binom{n}{1}(5/6)^{n-1}(1/6)^1 + \binom{n}{2}(5/6)^{n-2}(1/6)^2 + \binom{n}{3}(5/6)^{n-3}(1/6)^3 + \cdots$$

and, again by the Binomial Theorem,

$$(4/6)^n = (5/6-1/6)^n = \binom{n}{0}(5/6)^n(1/6)^0 - \binom{n}{1}(5/6)^{n-1}(1/6)^1 + \binom{n}{2}(5/6)^{n-2}(1/6)^2 - \binom{n}{3}(5/6)^{n-3}(1/6)^3 + \cdots$$

Subtract, and divide by 2. We obtain

$$(1/2)(1-(4/6)^n) = \binom{n}{1}(5/6)^{n-1}(1/6)^1 + \binom{n}{3}(5/6)^{n-3}(1/6)^3 + \binom{n}{5}(5/6)^{n-5}(1/6)^5 + \cdots .$$

The expression on the right, and therefore the one on the left, is our desired probability.