Problems, May 2011

Problem 1. Given six points in the *interior* of a circle of radius 1, show that the distance between two of these points is less than 1. This is obvious, maybe, but if it is one should be able to prove it.

Problem 2. Find all functions f from the reals to the reals such that

$$x(f(y))^2 + y(f(x))^2 = (x+y)f(x)f(y)$$

for all reals x, y. Prove that you have got them all.

Problem 3. Let a, b, and c be the roots of the cubic equation $x^3 + 3x^2 - 1 = 0$. Write down a cubic polynomial whose roots are a^2, b^2, and c^2. Finding a, b, and c may be difficult. Can we solve the problem without doing that?

Problem 4. A set of integers is called *double-free* if for any x in the set, $2x$ is not in the set. (a) What is the largest possible size of a double-free subset of $\{1, 2, 3, \ldots, 2011\}$? (The answer can be found without working too hard.) (b) How many double-free subsets of this size are there?