The Subgroup Structure and mod 2 Cohomology of O’Nan’s Sporadic Simple Group

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This is the first of two papers in which we determine the mod(2) cohomology of the sporadic group O’N of order $2^6 3^4 5^3 7^3 11 19 31 = 460,815,505,920$. O’N has 2-rank three, and, from the Gorenstein–Harada Theorem [GH], together with the results of [AMM, FM, M1, We], which determine the mod(2) cohomology rings of $J_1, M_{23}, G_2(q), D_8(q)$, with $q$ odd, there now remain only the Chevalley groups $U_3(8)$ and $S_5(8)$ among the 2-rank three simple groups for which the mod(2) cohomology ring has not been determined. In what follows we often use the Atlas notation [Co].

There are two maximal 2-local subgroups in O’N, $\text{Alp}_2^5 = 4^3 \cdot L_3(2)$, which is the non-split extension of the form

$$1 \rightarrow 4^3 \rightarrow \text{Alp}_2^5 \rightarrow L_3(2) \rightarrow 1$$

studied by Alperin [Alp], and the centralizer of an involution, the non-split extension $4 \cdot L_3(4):2_1$. They can be chosen to intersect in a subgroup $H_B = K_B \wr \mathcal{P}$ where $K_B = 4^3 \cdot 2^2$. In particular, their intersection contains a Sylow 2-subgroup. (The $B$ denotes a particular element of order three in the $\mathcal{P} \subset \text{Alp}_2^5$ which normalizes $K_B$.)

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In order to obtain cohomology results we have to get very precise information about the conjugacy classes of elementary 2-subgroups and their centralizers in O'N. Here is our first main result, sharpening results of O'Nan [ON] and Yoshiara [Yo]:

**Theorem A.** (a) There are two conjugacy classes of maximal elementary 2-groups in O'N, both isomorphic to $2^3$ with centralizers $I = 4^3$ and $II = 4 \times 2^3$ respectively.

(b) The Weyl groups of the two conjugacy classes$^1$ are the general linear group $L_3(2)$ for the first class and $\mathcal{C}_8$ for the second. In the second case there is a particular choice for the subgroups $4$ and $2^2$ so that the $4$ is fixed by $\mathcal{C}_8$ and, at the same time, there is a subgroup $\mathcal{C}_3 \subset \mathcal{C}_8$ which acts as $L_3(2)$ on the subgroup $2^2$.

**Remark.** (b) is contained in [ON, Yo], but (a) requires the very detailed information about the structure of the Alperin group $Alp_3^2$ contained in [Gr], and much of this paper is devoted to the proof of (a). Indeed, we completely determine the poset space $\mathcal{P}(O'N)$ of 2-elementary subgroups and inclusions and show in Section 2 that the quotient of its geometric realization under the action of O'N induced by conjugation has the form of a two-dimensional complex:

![Diagram](image)

The groups in boxes are the isotropy groups of faces and the labels in smaller type are the isotropy groups of edges.

In Section 4, using geometric interpretations of results of Webb [We], we derive a very useful consequence of the result above:

**Theorem B.** Let $\Gamma$ be the amalgamated product $Alp_3^2 *_{H}(4 \cdot L_3(4): 2_1)$ and $\pi: \Gamma \to O'N$ be the surjective homomorphism induced by sending $Alp_3^2$ and $4 \cdot L_3(4): 2_1$ isomorphically to the subgroups of O'N which intersect in

$^1$ Following the custom among topologists, if $H \subset G$ is an abelian subgroup of $G$ then $N_G(H)/C_G(H)$ is called the Weyl group of $H$ in $G$. Group theorists often call this the automizer of $H$ in $G$. 
\[ H^*_q; \text{ then the induced cohomology map} \]
\[ \pi^* : H^*(O^N; \mathbb{F}_2) \rightarrow H^*(\Gamma; \mathbb{F}_2) \]

is an isomorphism of rings.

Remark. This is very similar to the situations for \( M_{12} \), \( G_7(q) \), and \( ^3D_4(q) \) where similar surjective maps of amalgamated products induced isomorphisms in mod(2) cohomology. Similarly, a sporadic geometry enabled us to write the classifying space for \( M_{22} \) as a union of classifying spaces of three subgroups amalgamated over the classifying spaces of their respective intersections in [AM2]. However, in that case the generalized amalgamation turns out to be isomorphic to \( M_{22} \).

Next we turn to the structure of \( H^*(O^N; \mathbb{F}_2) \). In Section 1.6 we show that \( \text{Syl}_2(O^N) \) has seven conjugacy classes of \( 2^3 \)'s, and in Section 3 we begin the determination of the cohomology ring by describing the exact structure of the fusions of their centralizers in \( O^N \). Thus, we construct explicit fusions which fuse them into two conjugacy classes.

In Part II of [M2], it is shown that restriction to the seven centralizers of these \( 2^3 \)'s is injective in mod(2) cohomology for \( H^*(\text{Syl}_2(O^N); \mathbb{F}_2) \). Thus, using the Cartan–Eilenberg double coset formula, the elements in \( H^*(\text{Syl}_2(O^N); \mathbb{F}_2) \) which are in the image from \( H^*(O^N; \mathbb{F}_2) \) are precisely those elements whose images are fixed under the Weyl groups of the centralizers, as well as under the fusion identifications. In particular, we have

**Corollary C.** The two restriction maps in cohomology

\[ \text{res}^1 \oplus \text{res}^2 : H^*(O^N; \mathbb{F}_2) \rightarrow H^*(4^3; \mathbb{F}_2) \overset{L_3(2)}{\oplus} H^*(4 \times 2^2; \mathbb{F}_2) \]

together are injective in cohomology where \( I \) and \( II \) are the centralizers of the two conjugacy classes of \( 2^3 \)'s in \( O^N \) and are discussed in Theorem A.

The invariant substrings in Corollary C are given as follows. Recall that

\[ H^*(4^3; \mathbb{F}_2) = \mathbb{F}_2[\beta_2(1), \beta_2(2), \beta_2(3)] \otimes E(e_1(1), e_1(2), e_1(3)), \]

the tensor product of a polynomial algebra on two-dimensional generators and an exterior algebra on one-dimensional generators\(^2\) and that the \( e_1(l) \)'s and the \( \beta_2(l) \)'s are related by the \( \mathbb{Z}/4 \)-Bockstein, \( \beta_2(e_1(l)) = \beta_2(l) \), \( l = 1, 2, 3 \), which extends as a derivation to products. The action of \( L_3(2) \) is the usual one induced from the linear actions on the one- and two-dimen-\(^2\) A subscript on a generator in this paper will always denote its dimension.
sional generators, so from the results of [Mui] we have that the ring of invariants has the form

\[ \mathbb{F}_2 \langle d_n, d_{12}, d_{14} \rangle \left[ 1, E_3, \tilde{M}_1, \tilde{M}_n, \tilde{X}_7, \tilde{M}_{10}, \tilde{X}_{11}, \tilde{X}_{13} \right] \]

where the \( d_i \) are the Dickson elements in \( \mathbb{F}_2 \langle b_2(1), b_2(2), b_2(3) \rangle \),

\[
\begin{align*}
d_n &= b_2(1)^4 + b_2(2)^4 + b_2(3)^4 \\
&\quad + b_2(1)^2 b_2(2)^2 + b_2(1)^2 b_2(3)^2 + b_2(2)^2 b_2(3)^2 \\
&\quad + b_2(1) b_2(2) b_2(3) (b_2(1) + b_2(2) + b_2(3))
\end{align*}
\]

while

\[
\begin{align*}
d_{12} &= Sq^4(d_n), \quad d_{14} = Sq^2(d_{12}), \quad E_3 = e_1(1) e_1(2) e_1(3), \\
\tilde{M}_1 &= \beta_4(E_3) = b_2(1) e_1(2) e_1(3) + e_1(1) b_2(2) e_1(3) + e_1(1) e_1(2) b_2(3), \\
\tilde{M}_n &= Sq^2(\tilde{M}_2) = b_2(1)^2 e_1(2) e_1(3) + e_1(1) b_2(2)^2 e_1(3) \\
&\quad + e_1(1) e_1(2) b_2(3)^2, \\
\tilde{X}_7 &= \beta_4(\tilde{M}_n) = (b_2(1)^2 b_2(2) + b_2(1) b_2(2)^2) e_1(3) \\
&\quad + (b_2(1)^2 b_2(3) + b_2(1) b_2(3)^2) e_1(2) \\
&\quad + (b_2(2)^2 b_2(3)^2) e_1(1), \\
\tilde{M}_{10} &= Sq^4(\tilde{M}_n), \quad \tilde{X}_{11} = Sq^4(\tilde{X}_7), \quad \tilde{X}_{13} = Sq^2(\tilde{X}_{11}).
\end{align*}
\]

We now describe the structure of the invariants for II. Recall that \( H^*(G_1 \times G_1; \mathbb{F}_2) = H^*(G_1; \mathbb{F}_2) \otimes_p H^*(G_2; \mathbb{F}_2) \), \( H^*(G; \mathbb{F}_p) = \text{Hom}(G, \mathbb{F}_p) \), and

\[ H^*(2^i; \mathbb{F}_2) = \begin{cases} 
\mathbb{F}_2 \langle e_1 \rangle & \text{if } i = 1, \\
E(\lambda_i) \otimes \mathbb{F}_2 \langle b_2(\lambda) \rangle & \text{if } i \geq 2.
\end{cases} \]

Thus we have

\[ H^*(4 \times 2^2; \mathbb{F}_2) = \mathbb{F}_2 \langle b_2(\lambda), b_1, h_1 \rangle \otimes E(\lambda_1) \]

where \( \lambda \in \text{Hom}(4 \times 2^2, \mathbb{F}_2) \) is the unique non-trivial homomorphism with kernel \( 2^1 \subset 4 \times 2^2 \). The \( \mathcal{S} \) action is described in Theorem A, so if \( \triangleright \) is
the generator of order four fixed by $\mathcal{S}$ and $x, y$ are generators of order two, then the Klein group $K = \langle a, b \rangle < \mathcal{S}$ acts by $a(x) = x + 2r, a(y) = y, b(x) = x, b(y) = y + 2r$, while an $\mathcal{S}$; subgroup acts as $L_2(2)$ on $\langle x, y \rangle$. Consequently, the $\mathcal{S}$; invariants have the form

$$(\mathbb{F}_2[b_2(\lambda), b_1, h_1] \otimes E(\lambda_1))^\mathcal{S}; = \mathbb{F}_2[D_8, d_2, d_3] \otimes E(\lambda_1)$$

where $d_2 = b_2^2 + b_1h_1 + h_1^2$, $d_3 = b_2h_1 + b_1h_1^2$, and $D_8 = b_2(\lambda)^2 + d_2^2b_2(\lambda)^2 + d_2^3b_2(\lambda)$.

As a final step at this stage, one of the irreducible real representations of $O^\perp$ is used to determine the existence of a polynomial subalgebra $\mathbb{F}_2[d_8, d_{12}, d_{14}] \subset H^*(O^\perp; \mathbb{F}_2)$ and its image under restriction in $H^*(4^2; \mathbb{F}_2) \oplus H^*(4 \times 2^2; \mathbb{F}_2)$.

Next one applies the specific results of Part II of [M2] to determine the exact image of $H^*(O^\perp; \mathbb{F}_2)$ in $H^*(\text{Syl}_3(O^\perp); \mathbb{F}_2)$. We summarize the table of generators and their restrictions to the seven conjugacy classes of centralizers of $2^3$'s in Section 3, and from this we can read off the generators of $H^*(O^\perp; \mathbb{F}_2)$.

The description of the ring $H^*(O^\perp; \mathbb{F}_2)$ is given as follows. It has 12 generators, one in each dimension from 3 to 14: $d_3, M_4, Y_5, M_6, Y_7, d_8, Y_9, M_{10}, X_{11}, d_{12}, X_{13}$, and $d_{14}$ with $M_4, M_6, M_{10}$ as the generators of the radical. The action of the Steenrod algebra on these generators is given by

$Sq^2d_3 = Y_5,$
$Sq^2(Y_5) = Y_9,$
$Sq^2M_4 = M_6,$
$Sq^2(M_6) = M_{10},$
$Sq^2(X_5) = X_{11},$
$Sq^2(X_{11}) = X_{13},$
$Sq^1(d_8) = d_{12},$
$Sq^2(d_{12}) = d_{14} = d_5(d_3d_8 + X_{11}),$

and the action of $Sq^1$ on the generators is given by the formulae

$Sq^1(Y_5) = d_5^2,$
$Sq^1(M_6) = d_5M_4,$
$Sq^1(Y_9) = d_5X_7,$
$Sq^1(M_{10}) = M_4X_2,$
$Sq^1(X_{13}) = d_5X_{11}.$

$Sq^1$ is zero on the generators which are not listed. Also, there are a number of basic relations such as

$d_5d_{14} = Y_5d_{14} = Y_9d_{14} = 0,$

$Y_5^2 = d_5X_7,$

$Y_9^2 = d_5(X_7d_4 + d_3d_{12} + d_3^2),$

$X_7^2 = d_5X_{11}, X_{11}^2 = d_5(X_{11}d_4 + X_7d_{12} + X_7d_3^2).$

The best way to understand the structure of this algebra is to explain its embedding into $H^*(4^2; \mathbb{F}_2) \oplus H^*(4 \times 2^2; \mathbb{F}_2)^\mathcal{S};$. The map on genera-
tors is given by
\[ d_3 \rightarrow (0, d_3), \quad X_7 \rightarrow (\tilde{X}_7, d_2^3 d_3), \quad X_{11} \rightarrow (\tilde{X}_{11}, d_2^3 d_3), \]
\[ M_4 \rightarrow (\tilde{M}_4, \lambda d_3), \quad d_8 \rightarrow (d_8, D_8 + d_1^3), \quad d_{12} \rightarrow (d_{12}, d_2^3 D_8 + d_1^3), \]
\[ Y_5 \rightarrow (0, d_2^3 d_3), \quad Y_9 \rightarrow (0, d_2^3 d_3), \quad X_{13} \rightarrow (\tilde{X}_{13}, d_2^3 d_3), \]
\[ M_6 \rightarrow (\tilde{M}_6, \lambda d_2^3 d_3), \quad M_{10} \rightarrow (\tilde{M}_{10}, \lambda d_2^3 d_3), \quad d_{14} \rightarrow (d_{14}, 0). \]

Using this embedding of the algebra it is easy to verify the relations and squaring operations above, as well as to write down the remaining relations. One can, for example, verify that the image of the subalgebra \( \mathbb{F}_2[d_3, d_8, d_{12}](1, Y_5, X_7, Y_9, X_{11}, X_{13}) \) in the second factor is an injection. In particular, the image is free over the polynomial algebra on \( d_1, d_8, d_{12} \).

Moreover, if we look at the part generated by \( M_4, M_6, \) and \( M_{10} \) over this algebra, which we denote \( \mathcal{A} \), we get a short exact sequence of modules
\[ 0 \rightarrow \mathbb{F}_2[d_3, d_8, d_{12}](1, Y_5, X_7, Y_9, X_{11}, X_{13}) M_4 \rightarrow \mathcal{A} \rightarrow \mathbb{F}_2[d_8, d_{12}](M_6, M_{10}) \rightarrow 0. \]

But the extension is non-trivial, \( d_3 M_6 = Y_5 M_4, \ d_3 M_{10} = Y_9 M_4 \), as is quickly verified by checking restrictions—for example, the restrictions of both \( d_3 M_{10} \) and \( Y_6 M_4 \) are \((0, \lambda d_2^3 d_3)\)—so we can rewrite \( \mathcal{A} \) in the form
\[ \mathcal{A} = \mathbb{F}_2[d_3, d_8, d_{12}](M_4, M_6, M_{10}, X_7 M_4, X_9 M_4, X_{11} M_4, X_{13} M_4). \]

Similarly, if we look at the image in the first factor, it has the form
\[ \mathbb{F}_2[d_8, d_{12}, d_{14}](1, M_4, \tilde{M}_6, \tilde{X}_7, M_{10}, \tilde{X}_{11}, \tilde{X}_{13}, d_{14} E_3) \]
where the last factor, \( d_{14} E_3 \), is the image of \( X_7 M_{10} = X_{11} M_6 = X_{13} M_4 \).

Remark. Quillen proved that the minimal primes in \( H^*(G; \mathbb{F}_p) \) are in one-to-one correspondence with the conjugacy classes of maximal elementary \( p \)-groups in \( G \), the correspondence being given by associating to each such group the kernel of the restriction map in cohomology to the polynomial subring of \( H^*(E_p; \mathbb{F}_p) \) [AM, Chap. IV]. In the case of O’N there are exactly two minimal primes, corresponding to the two \( Z/p^2 \)'s, \( \mathcal{P}_1 \) and \( \mathcal{P}_{11} \). The quotient by the first is simply \( \mathbb{F}_2[d_3, d_8, d_{12}] \); but the quotient by the second is
\[ \mathbb{F}_2[d_3, d_8, d_{12}](1, Y_5, X_7, Y_9, X_{11}, X_{13}). \]

\(^3\) When we write for example \( \mathcal{A}(A, B, C) \), this denotes an algebra which is free as a module over \( \mathcal{A} \) on the three generators \( A, B, C \).
and this integral domain is not closed in its quotient field since \( d_2 \) is integral over it, being a root of the monic polynomial \( x^6 + d_4 x^5 + (d_{12} + d_4^2) = 0 \), but is not contained in it.

\( H^*(\mathbb{O}'N; \mathbb{F}_2) / \mathcal{P}_{11} \) is the first example with this property that is known to us.

**Remark.** The \( Sq^1 \) structure described above, following from the map to \( H^*(4 \times 2^2; \mathbb{F}_2)^{\times 4} \), gives the \( E_2 \) term of the Bockstein spectral sequence which explores the higher 2-torsion in \( H^*(\mathbb{O}'N; \mathbb{Z}) \) as the cohomology of \( H^*(\mathbb{O}'N; \mathbb{F}_2) \) with respect to \( Sq^1 \). (This makes sense since \( (Sq^1)^2 = 0 \).) The result is

\[
E_2 = \mathbb{F}_2[d_6, d_{12}](1, d_3, M_4, X_7, X_{11}, X_{11}, M_4)
\]

\[
\oplus \mathbb{F}_2[d_6, d_{12}, d_{14}][d_{14}](1, M_4, M_6, X_7, M_{10}, X_{11}, X_{11}, X_{13}, M_4, X_{11}).
\]

where the second line is the contribution from the group \( H^*(4^2; \mathbb{F}_2) \) and the first line comes from \( H^*(4 \times 2^2; \mathbb{F}_2) \). Next, using our knowledge of the \( \mathbb{Z}/4 \) Bockstein in \( H^*(4^2; \mathbb{F}_2) \) we see that \( E_3 \), which is given as the homology of \( E_2 \) with respect to this \( \mathbb{Z}/4 \)-Bockstein, comes from the following differentials: \( \beta_2(d_{14}^2E_2) = d_{14}^2 M_4 \), \( \beta_2(d_{14} M_8) = d_{14} X_7 \), \( \beta_2(d_{14} M_{10}) = d_{14} X_{11} \), and \( \beta_2(d_{14} X_{13}) = d_{14}^2 \), while \( d_{14} \) either goes to zero or \( X_{11} M_4 \). It follows that the \( E_3 \) term has the form

\[
\mathbb{F}_2[d_6, d_{12}](1, d_3, M_4, X_7, X_{11}, d_{14}, X_{11}, M_4, d_{14} M_4),
\]

or

\[
\mathbb{F}_2[d_6, d_{12}](1, d_3, M_4, X_7, X_{11}, d_{14}, M_4).
\]

We do not know the structure of the higher differentials, but it is clear how they must work: the role of \( X_7 \) is to hit \( d_8 \), that of \( X_{11} \) is to hit \( d_{12} \), \( d_3 \) must hit \( M_4 \), and \( d_{14} M_4 \) must hit \( X_{11}d_6 + X_7d_{12} \). The only question is at what level these differentials appear.

The Poincaré series for \( H^*(\mathbb{O}'N; \mathbb{F}_2) \) is given as

\[
\frac{(1 + x^4)(1 + x^4 + x^7 + x^9 + x^{11} + x^{13})}{(1 - x)(1 - x^3)(1 - x^5)(1 - x^9)(1 - x^{12})} + \frac{x^6 + x^{10}}{(1 - x^8)(1 - x^{12})} + \frac{x^{14}(1 + x^4 + x^6 + x^{10} + x^7 + x^{11} + x^{13} + x^{17})}{(1 - x^8)(1 - x^{12})(1 - x^{14})}
\]

where the third term corresponds to the elements in the minimal prime \( \mathcal{P}_{11} = (d_{14}) \). The rational function above expands out as

\[
x^{24} + x^{31} + x^{20} + x^{19} + 2x^{18} + x^{17} + x^{16} + 3x^{15} + 2x^{14} + 2x^{13} + 2x^{12} + x^{11} + 2x^{10} + x^9 + x^8 + x^7 + 2x^6 + x^5 + x^4 + x^3 + 1
\]

\[
(1 - x^7)(1 - x^8)(1 - x^{12})
\]
and, with denominator \((1 - x^8)(1 - x^{12})(1 - x^{14})\), as
\[
\frac{f(x)}{(1 - x^8)(1 - x^{12})(1 - x^{14})}
\]
where \(f(x)\) is the symmetric polynomial of degree 31,
\[
x^{31} + x^{28} + x^{27} + x^{26} + 2x^{25} + 2x^{24} + x^{23} + 3x^{22} + 3x^{21} + 3x^{20} + 3x^{19} \\
+ 4x^{18} + 3x^{17} + 4x^{16} + 4x^{15} + 3x^{14} + 4x^{13} + 3x^{12} + 3x^{11} \\
+ 3x^{10} + 3x^9 + x^8 + 2x^7 + 2x^6 + x^5 + x^4 + x^3 + 1. \quad (\ast)
\]
The reason for the expression \((\ast)\) above is that we believe \(H^*(O'N; \mathbb{F}_2)\) is freely and finitely generated over \(\mathbb{F}_2\)[\(d_4, d_12, d_{14} + w\)], for some \(w\) contained in the ideal \((d_4, Y_3, Y_4, \ldots)\) of elements which project trivially into \(H^*(4^3; \mathbb{F}_2)\), so \((\ast)\) gives a specification of the dimensions in which the generators should occur. Moreover, \((\ast)\) indicates the strong likelihood of the existence of a Poincaré duality space or manifold of dimension 31 which reflects many of the properties of \(O'N\) if, indeed, \(H^*(O'N; \mathbb{F}_2)\) is freely and finitely generated over the polynomial ring above.

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1. THE 2-LOCAL STRUCTURE OF \(O'N\)

Denote by \(\text{Alp}_n^1\) the split extension
\[
1 \to (2^n)^3 \to \text{Alp}_n^1 \to L_3(2) \to 1 \quad (1.1)
\]
and by \(\text{Alp}_n^2\) the non-split extension
\[
1 \to (2^n)^3 \to \text{Alp}_n^2 \to L_3(2) \to 1
\]
where in both cases \(L_3(2) = GL_3(\mathbb{F}_2)\) acts on \((2^n)^3\) in the usual way. These are the Alperin groups first introduced in [Alp], which are uniquely determined and play a critical role in the classification theorem for sporadic simple groups. Specifically, \(\text{Alp}_n^1\) is a maximal subgroup of \(\omega_4 \cong L_3(2)\) of odd index. Also, \(\text{Alp}_n^2\) is a maximal subgroup of \(G_2(q), \quad \text{Sp}_6(q)\) for \(q\) odd, and is of odd index if \(q \equiv 3, 5 \mod 8\). Let HS denote the Higman–Sims sporadic group. Then \(\text{Alp}_n^1\) is maximal and of odd index in HS. Additionally, the Sylow 2-subgroup of \(\text{Alp}_n^2\) is the Sylow 2-subgroup of the Mathieu group \(M_{12}\).
At this point the obvious question arises: does there exist a finite simple group \( G \) with \( \mathrm{Syl}_2(G) = \mathrm{Syl}_2(\text{Alp}_2^2) \)? O'Nan [ON] gave a positive answer to this question, characterizing the sporadic group \( O'N \) as the unique simple group having this Sylow 2-subgroup. Its existence and uniqueness were later verified by Andrelli and Sims [A]. O'N has order \( 2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31 = 460,815,505,920 \).

In this section, we describe the 2-local structure of \( O'N \), as this is what we need for the cohomology calculations. The following proposition combines results of [ON, Yo].

**Proposition 1.1.** (1) \( O'N \) has only one conjugacy class of involutions \( \langle j \rangle \). The centralizer of \( j \) is the group \( 4 \cdot L_3(4).2 \).

(2) \( O'N \) has exactly two conjugacy classes of maximal subgroups of odd index \( \text{Alp}_2^2 = 4 \cdot L_3(2) \) and \( 4 \cdot L_3(4).2 \).

We will show shortly that the two subgroups above, \( \text{Alp}_2^2 \) and \( 4 \cdot L_3(4).2 \), intersect in a common subgroup \( H_2 \) of order \( 2^9 \cdot 3 \), so \( O'N \) contains the configuration

\[
4 \cdot L_3(4) : 2_1 \cdot \text{Alp}_2^2 \cdot \cdot \cdot
\]

which completely controls the cohomology of \( O'N \).

**The Sylow 2-Subgroup of \( O'N \)**

The group \( \mathrm{Syl}_2(i) \) (\( i = 1 \) or 2) has a presentation with five generators \( v_1, v_2, v_3, s, \) and \( t \), so that \( \langle v_1, v_2, v_3 \rangle = (\mathbb{Z}/4)^3 \), \( t^2 = 1 \), and the remaining relations are given as

\[
sv_1s^{-1} = v_2, \quad sv_2s^{-1} = v_3, \quad sv_3s^{-1} = v_1v_2^{-1}v_3, \\
tv_1t = v_3^{-1}, \quad tv_2t = v_3^{-1}, \quad tv_3t = v_1^{-1}, \quad tst = s^{-1}
\]

\[
s^4 = \begin{cases} v_1 & \text{in the case of } \mathrm{Syl}_2(\text{Alp}_2^2), \\
1 & \text{in the case of } \text{Alp}_2^1. \end{cases}
\]

It is pointed out in [Alp] that \( \mathrm{Syl}_2(\text{Alp}_2^2) \) is a rank 3 group. To obtain the structure of the Sylow 2-subgroup of \( \text{Alp}_2^2 \) for \( n > 2 \), \( i = 1, 2 \), nothing changes in the presentation above except that \( v_1, v_2, \) and \( v_3 \) now generate a group \( (2^n)^3 \).

We also need the structure of the conjugacy classes of elements of order two, and the elementary two groups in \( \mathrm{Syl}_2(\text{Alp}_2^2) \). The following two results are from [ON].
Lemma 1.4. There are seven conjugacy classes of involutions in \( \text{Syl}_1(\text{Alp}_2^3) \). Representatives and the centralizers for these representatives are given in the following table.

<table>
<thead>
<tr>
<th>Class</th>
<th>Representative</th>
<th>Number in class</th>
<th>Centralizer</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( v_1^2 v_3^2 )</td>
<td>1</td>
<td>( \langle v_1, v_2, v_3, s, t \rangle )</td>
</tr>
<tr>
<td>B</td>
<td>( v_1^2 v_2^2 )</td>
<td>2</td>
<td>( \langle v_1, v_2, t, s^2, v_1^{-1} \rangle )</td>
</tr>
<tr>
<td>C</td>
<td>( v_2^2 )</td>
<td>4</td>
<td>( \langle v_1, v_2, v_1^{-1}, t \rangle )</td>
</tr>
<tr>
<td>D</td>
<td>( s^2 v_1^{-1} )</td>
<td>16</td>
<td>( \langle v_1 v_3, v_1^2 v_3^2, s^2, v_1^{-1} \rangle )</td>
</tr>
<tr>
<td>E</td>
<td>( st )</td>
<td>32</td>
<td>( \langle v_1 v_3, v_1^2 v_3^2, s \rangle )</td>
</tr>
<tr>
<td>F</td>
<td>( t )</td>
<td>16</td>
<td>( \langle v_1 v_3, v_1^2 v_3^2, s^2, v_1^{-1}, t \rangle )</td>
</tr>
<tr>
<td>G</td>
<td>( w_2 )</td>
<td>16</td>
<td>( \langle v_1 v_3, v_2^2 v_3, w_2 \rangle )</td>
</tr>
</tbody>
</table>

Lemma 1.5. (i) The centralizer of \( s^2 v_1^{-1} = \langle v_1 v_3, v_1^2 v_3^2, s^2, v_1^{-1}, t \rangle \) is isomorphic to \( 2 \times (4 \ast D_8) \) where \( 4 \ast D_8 \) is the central product.

(ii) The centralizer of \( t = \langle v_1 v_3, v_1^2 v_3^2, s^2, v_1^{-1}, t \rangle \) is also isomorphic to \( 2 \times (4 \ast D_8) \).

(iii) The centralizer of \( w_2 = \langle v_1 v_3, v_1^2, s^2 v_1 v_2 v_3, w_2 \rangle \) is again isomorphic to \( 2 \times (4 \ast D_8) \).

(iv) The centralizer of \( st = \langle v_1 v_3, v_1^2, s t \rangle \) is isomorphic to \( 4 \times (2)^2 \).

From these lemmas it is direct to enumerate the conjugacy classes of elementary 2-groups \( 2^3 \subset \text{Syl}_1(\text{Alp}_2^3) \). We have

Corollary 1.6. There are seven conjugacy classes of elementary 2-groups

\[ 2^3 \subset \text{Syl}_1(\text{Alp}_2^3). \]

They have representatives as follows: from the centralizer of \( v_2^2 \),

\[ \Gamma = \langle v_1^2, v_2^2, v_3^2 \rangle, \quad \Gamma' = \langle (v_1 v_3)^2, v_2^2, t \rangle, \quad \Gamma'' = \langle (v_1 v_3)^2, v_2^2, v_2 t \rangle; \]

from the centralizer of \( s^2 v_1^{-1} \),

\[ \Gamma'' = \langle (v_1 v_3)^2, (v_1 v_2)^2, s t \rangle; \]

and from the centralizer of \( s^2 v_1^{-1} \),

\[ \Gamma' = \langle (v_1 v_2)^2, (v_1 v_2)^2, s^2 v_1^{-1} \rangle, \quad \Gamma'' = \langle (v_1 v_2)^2, v_1 v_3 t, s^2 v_1^{-1} \rangle, \]

\[ \Gamma''' = \langle (v_1 v_2)^2, t, s^2 v_1^{-1} \rangle. \]

Proof. The listing of the possible conjugacy classes of \( 2^3 \)-subgroups is routine when we note that each subgroup will contain \( (v_1 v_3)^2 \), and if it also contains \( (v_1 v_2)^2 \) then it must contain an element in one of the remaining
five conjugacy classes. Consequently we need only list the $2^3$'s in the
centralizers of these five elements and cancel those that are conjugate.
This is aided when we note that $t s^2 v_1^{-1} = s t v_2 s^{-1}$, which directly implies
that the three $2^3$'s which arise from the centralizer of $t v_2$ are listed in the
groups for $v_2^2$ and $s^2 v_1^{-1}$. It is also helpful to note that the group
$\langle (v_1 v_3)^2, s^2 v_1 v_3^{-1}, t \rangle$ which occurs in the centralizer of $t$ is conjugate to
$\langle (v_1 v_3)^2, v_1 v_3 t, s^2 v_1 v_3^{-1} \rangle$ under the action of $s^2$.

Remark. We use the primes in our labeling of the conjugacy classes of
elementary 2-subgroups above, since we reserve the unprimed groups
I, II, ..., VII to denote the centralizers of the groups above. Thus $I = \langle v_1, v_3, v_3^{-1} \rangle$, $II = \langle v_1 v_3^{-1}, v_2, t \rangle$, and so on. Except for I, each of the
centralizers is a copy of $4 \times 2^2$.

The following result is very helpful in analyzing $H^*(\text{Syl}_2(O'N); F_2)$ since
the cohomology rings of the dihedral groups, $D_{2^n}$, and wreath products
are well understood [AM, Chap. IV].

**Proposition 1.7.** The subgroup $\langle v_1 v_3 \rangle = 4$ is normal in $\text{Syl}_2(\text{Alp}_2^2)$ with quotient

$$
\Gamma_2 = \text{Syl}_2(\text{Alp}_2^2)/\langle v_1 v_3 \rangle \cong D_8 \times 2
$$

where $D_8$ is the dihedral group of order 8.

**Proof.** $\langle v_1 v_3 \rangle \triangleleft \text{Syl}_2(\text{Alp}_2^2)$ since $s v_1 v_3 s^{-1} = v_1 v_3$, $t v_1 v_3 = (v_1 v_3)^{-1}$.
The quotient is obtained by adjoining the relation $v_3 = v_1^{-1}$. In particular,
s$^2 = 1$ so the projection $\Gamma_2 \to D_8$ sending $v_i$ to 1 for $i = 1, 2$ is split, and
we can write $\Gamma_2$ as the semi-direct product

$$
\Gamma_2 = 4^2 : D_8.
$$

We can write $D_8 = 2 \times 2$ with $\langle t, s^2 t \rangle$ being the $2^2 \subset 2^2$: $2 = 2 \times 2$ while the element acting to interchange them is $st$. Also, $t$ commutes with $v_1$ in $\Gamma_2$ while $w_2 t = v_1^{-1}$, $s^2 w_2 t s^{-2} = s^2 v_1 s^{-2} = v_3 = v_1^{-1}$, and $s^2 w_2 t s^{-2} = s^2 v_1 s^{-2} = v_2$, so

$$
4^2 : \langle t, s^2 t \rangle = D_8 \times D_8 = \langle v_1, t \rangle \times \langle v_1, s^2 t \rangle.
$$

Finally, $(st)^2 = 1$ and

$$
st v_1 s t^{-1} = s v_1^{-1} s^{-1} = v_1^{-1} = v_1,
$$

so $st$ acts to interchange the two copies of $D_8$ above and the proposition
follows.
Remark. The group $\text{Alp}_{2}^{2}$ is not the centralizer of an involution in $O'N$; that is the extension $4 \cdot L_{3}(4): 2_{1}$. In particular, from [ON] we have that the quotient group $L_{3}(4): 2$ of the involution centralizer has the group $\text{Syl}_{2}(\text{Alp}_{2}^{2})/\langle v_{1}v_{3}^{2}v_{3}^{-1} \rangle$ as its Sylow 2-subgroup, and this group is not the quotient analyzed above nor is it isomorphic to it. Indeed, [ON] shows that there are precisely two conjugacy classes of elements of order 4 in $O'N$, the first represented by $v_{1}v_{3}$ and the second by $v_{1}v_{3}^{2}v_{3}^{-1}$, and it is this second which represents the 4 in $4 \cdot L_{3}(4): 2_{1}$.

Remark. In the table in Section 3 where the generators of $H^*(\text{Syl}_{2}(O'N); \mathbb{F}_{2})$ are listed, the elements $e_{x}, e_{y}, e_{z}, I(x), a_{x}, V_{3}$, and $I(\omega)$ all come from $H^*(D_{8} \wr 2; \mathbb{F}_{2})$ under the induced cohomology map for the projection above.

The Explicit Structure of $\text{Alp}_{2}^{2}$

Partly in order to understand the fusion in $O'N$, but also to understand the configuration 1.2, we now give an explicit set of generators for $\text{Alp}_{2}^{2}$ following an article by Griess [Gr], where he constructs $\text{Alp}_{2}^{2}$ as a subgroup of $\text{Aff}(\mathbb{Z}/8) \subset \text{GL}_{3}(\mathbb{Z}/8)$ consisting of matrices of the form $\begin{pmatrix} 1 & 0 \\ W & V \end{pmatrix}$, where $W \in \text{GL}_{3}(\mathbb{Z}/8)$ and $V$ is a $3 \times 1$-column vector. In particular, the subgroup $\langle v_{1}, v_{3}, v_{3} \rangle$ is given as the subgroup of $\text{GL}_{3}(\mathbb{Z}/8)$ of all matrices of the form $\begin{pmatrix} 1 & 0 \\ V & I_{3} \end{pmatrix}$, where $I_{3}$ is the $3 \times 3$ identity matrix and $V$ is any $3 \times 1$ column vector with even entries.

The upper-triangular matrices in $\text{GL}_{3}(2)$ give a copy of the 2-Sylow subgroup so we look in those tables for elements $W$ which when reduced mod(2) give upper triangular matrices. Explicit generators for the 2-Sylow subgroup corresponding to the generators we have been studying can be given as

$$
t \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 7 & 1 & 2 \\ 2 & 0 & 1 & 4 \\ 7 & 0 & 0 & 7 \end{pmatrix}, \quad s \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 7 & 6 \\ 2 & 6 & 1 & 7 \\ 5 & 4 & 4 & 7 \end{pmatrix}.
$$

With respect to these choices for $s, t$, we find that a consistent choice for $v_{1}$ is

$$
v_{1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 6 & 0 & 1 & 0 \\ 6 & 0 & 0 & 1 \end{pmatrix}.
$$
so, since \( t_2 = s t_1 s^{-1} \), \( t_3 = s^2 t_1 s^{-2} \), we find that representatives for the generators \( t_2, t_3 \) are given by

\[
\begin{align*}
\psi_2 &\leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}, & \psi_3 &\leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 6 & 0 & 0 & 1 \end{pmatrix}.
\end{align*}
\]

We record the representations of the elements \( ts, s^2, \) and \( ts^2 \):

\[
\begin{align*}
ts &\leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 5 & 2 & 7 \\ 0 & 6 & 1 & 3 \\ 2 & 4 & 4 & 1 \end{pmatrix}, & s^2 &\leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 6 & 1 \\ 3 & 0 & 7 & 4 \\ 0 & 0 & 4 & 5 \end{pmatrix}, \\
ts^2 &\leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 5 & 1 & 5 \\ 5 & 0 & 7 & 0 \\ 7 & 0 & 4 & 3 \end{pmatrix}.
\end{align*}
\]

Next we need two elements, \( \alpha, \beta \), of order 3, for which the \( W \) terms restrict mod(2) to

\[
\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

and

\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}
\]

respectively. These matrices \( \alpha, \beta \), together with the elements above, must generate the entire group \( \text{Alp}_3^1 \). Suitable choices are

\[
\alpha \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 7 & 4 & 3 & 0 \\ 4 & 1 & 3 & 2 \\ 0 & 4 & 0 & 1 \end{pmatrix}, \quad \beta \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 7 & 0 & 4 & 3 \\ 6 & 2 & 1 & 3 \end{pmatrix}.
\]

Then

\[
\alpha^{-1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 7 & 3 & 5 & 6 \\ 7 & 7 & 4 & 0 \\ 4 & 4 & 4 & 1 \end{pmatrix}, \quad \beta^{-1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 4 & 4 \\ 5 & 6 & 3 & 5 \\ 7 & 0 & 7 & 4 \end{pmatrix}.
\]

\footnote{This model for \( \text{Alp}_3^1 \) is not quite the model in [ON], as it reflects the topologists' preference for left actions instead of right actions. In particular, in this paper conjugation of \( x \) by \( a \) is \( a x a^{-1} \) and not, as is more usual with group theorists, \( a^{-1} x a \). We also use the notation \( \alpha(x) \) for \( a x a^{-1} \).}
Lemma. 1. Let $K_n = \langle v_1^3, v_2^3, v_3^3, t, s, t^2 \rangle$. Then $\alpha$ normalizes $K_n$ with action $\alpha(s^2) = (v_3 v_2^{-1}) v_3 s^2$, $\alpha(t^2) = (v_1 v_3^{-1}) v_2 s^2$, $\alpha(v_1) = v_1 (v_2 v_3^{-1})^{-1}$, $\alpha(v_2) = v_1^2 v_2^{-1}$, and $\alpha(v_3) = v_1^2 v_3^{-1}$. Also $\tau \alpha t = \alpha^{-1}$.

2. Let $K_n = \langle v_1, v_2, v_3, t, s, t^2 \rangle$. Then $\beta$ normalizes $K_n$ with action $\beta(t) = v_3^{-1} t s^2$, $\beta(s^2) = (v_3 v_2^{-1})^{-1}$, $\beta(v_1) = v_1 v_2^{-1}$, $\beta(v_2) = v_3^{-1}$, and $\beta(v_3) = v_2 v_3^{-1}$, while $t s^2 \beta t s^2 = \beta^{-1}$.

Proof. This is a direct calculation. For example, we have

$$\alpha s^2 \alpha^{-1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 7 & 4 & 3 & 0 \\ 4 & 1 & 3 & 2 \\ 0 & 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 6 & 1 \\ 3 & 0 & 7 & 4 \\ 0 & 0 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 7 & 3 & 5 & 6 \\ 7 & 7 & 4 & 0 \\ 4 & 4 & 1 & 1 \end{pmatrix}$$

and, in turn, this is seen to represent $v_1^3 v_2^3 v_3^3 t s^3$. Next we find

$$\alpha t s \alpha^{-1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 6 & 3 & 6 & 1 \\ 7 & 0 & 7 & 4 \\ 6 & 0 & 4 & 5 \end{pmatrix}$$

which is the image of $(v_1 v_3)^3 v_2 s^2$. Also, it is direct to check that $t \alpha t = \alpha^{-1}$, and we have

$$\alpha v_1 \alpha^{-1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 6 & 0 & 0 & 1 \end{pmatrix}, \quad \alpha v_2 \alpha^{-1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}, \quad \alpha v_3 \alpha^{-1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 6 & 0 & 0 & 1 \end{pmatrix}.$$  

Consequently, $\alpha v_1 \alpha^{-1} = v_1 v_3^2 v_2^3$, $\alpha v_2 \alpha^{-1} = v_1^2 v_3^2$, and $\alpha v_3 \alpha^{-1} = v_1^2 v_2^3$. Thus, the action of $\alpha$ on the subgroup of $Syl_2(Alp_2)$ which it normalizes is completely determined.
We also check directly that $ts^3\beta ts^3 = \beta^{-1}$. It remains to determine the action of $\beta$ on the subgroup it normalizes. We have

$$\beta t \beta^{-1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 5 & 1 & 5 \\ 3 & 0 & 7 & 0 \\ 1 & 0 & 4 & 3 \end{pmatrix}, \quad \beta s^3 \beta^{-1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 7 & 1 & 2 \\ 4 & 0 & 1 & 4 \\ 7 & 0 & 0 & 7 \end{pmatrix},$$

while

$$\beta v_1 \beta^{-1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix}, \quad \beta v_2 \beta^{-1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 6 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}, \quad \beta v_3 \beta^{-1} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix}.$$

Thus we have the action of $\beta$ on $\langle t, s^2, v_1, v_2, v_3 \rangle$: first, $\beta v_1 \beta^{-1} = v_1 v_2^{-1}$, and $\beta v_2 \beta^{-1} = v_1^{-1}$, $\beta v_1 \beta^{-1} = v_2 v_3^{-1}$, second, $\beta t \beta^{-1} = v_1 t s^2$; and third, $\beta s^3 \beta^{-1} = (v_2 v_3)^{-1} t$. 

**Remark.** The action of $\alpha$ on $\langle v_1, v_2, v_3 \rangle$ splits as a trivial action on $4 = \langle v_1 v_2 v_3^2 \rangle$, and an action on $4^2 = \langle v_1 v_2^{-1} v_3^2, v_1 v_3^{-1} \rangle$ where

$$\alpha v_1^{-1} v_2^{-1} v_3 \alpha^{-1} = v_1 v_2 v_3^{-1}, \quad \alpha v_1^{-1} v_2^{-1} v_3 \alpha^{-1} = \left( (v_1 v_2^{-1} v_3)(v_2 v_3^{-1}) \right)^{-1}.$$

We have $\langle t, \alpha \rangle \cong \mathcal{S}_3$, and

$$H_\alpha = \langle t, \alpha, ts^2, v_1, v_2, v_3 \rangle \cong K_\alpha : \mathcal{S}_3. \tag{1.9}$$

Similarly, $\langle ts^3, \beta \rangle \cong \mathcal{S}_3$, so

$$H_\beta = \langle ts^3, \beta, t, s^2, v_1, v_2, v_3 \rangle \cong K_\beta : \mathcal{S}_3. \tag{1.10}$$

Also, for both $H_\alpha, H_\beta$, the quotients by $\langle v_1, v_2, v_3 \rangle$ are copies of $\mathcal{S}_3$. Note in particular that $\beta v_1 v_2 \beta^{-1} = v_1 v_3^2$ so $\beta(v_1 v_2)^{-1} = (v_1 v_3)^2$ and, since $4 \cdot L_3(4) \cdot 2$ is the centralizer of $v_1 v_3^2$ in $O' \mathcal{N}$, we have

$$\text{Alp}_2^3 \cap 4 \cdot L_3(4) \cdot 2 = H_\beta \cong 4^3 : \mathcal{S}_3.$$

**Remark 1.11.** When we project $K_\alpha$ to $\text{Syl}_2(O' \mathcal{N})/\langle v_1 v_3 \rangle$, we get that $s^2$ acts to invert every element of $\langle v_1, v_2, v_3 \rangle/\langle v_1 v_3 \rangle \cong 4^2$, while $ts$ acts to
exchange the two generators. But this is the description of $\text{Syl}_2(M_{12})$ as given, for example, in [FM]. This observation will be crucial in the calculation of $H^\bullet(\text{Syl}_2(O'N); F_2)$ given in [M2].

A Key Fusion in $\text{Alp}_2^3$

In Section 2 we will see that there are exactly two conjugacy classes of rank 3 elementary abelian 2-subgroups in $O'N$. For this we will need the following lemma describing a specific fusion in $\text{Alp}_2^3$.

**Lemma 1.12.** In $\text{Alp}_2^3$ the groups $\langle (v_1v_3)^2, (v_1v_2)^2, s^2v_1^{-1} \rangle$ and $\langle (v_1v_3)^2, (v_1v_2)^2, st \rangle$, both isomorphic to $2^3$, are conjugate.

**Proof.** Take the element $\alpha \in \text{Alp}_2^3$ of order 3 described in (1.8). Then

$$\alpha(v_1v_3)^2\alpha^{-1} = (v_1^2 v_2^2 v_3^2)(v_1^2) = (v_2v_3)^2$$

$$\alpha(v_1v_2)^2\alpha^{-1} = (v_1^2 v_2^2 v_3^2)(v_2^2) = (v_1v_3)^2$$

$$\alpha(s^2v_1^{-1})\alpha^{-1} = v_1^{-1}v_2^{-1}v_3ts^2(v_1^{-1}v_2^{-1})^{-1}.$$  

Now $v_1^{-1}v_2^{-1}(v_1t) = v_1^{-1}(v_2^{-1}t)v_1^{-1} = (v_1^{-1}t)v_2v_1^{-1}$, and

$$s^3(v_1^{-1}v_2v_3) = (v_2(v_1v_3)^{-1}v_1v_2)s^3 = (v_1v_2^2)(v_1v_3)^{-1}s^3 = (v_1v_2^2)s^{-1}$$

so

$$\alpha(s^2v_1^{-1})\alpha^{-1} = v_1^{-1}v_2^{-1}v_3ts^2v_1^{-1}v_2v_3$$

$$= t(v_2v_3v_1^{-1})v_1v_2^2s^{-1} = w_3v_2^{-1}s^{-1}$$

$$= ts^{-1}(v_2^{-1}v_1) = sw_3v_2^{-1}$$

$$= v_2v_3ts^3.$$  

Now we have

$$(v_1v_2v_3)s^3(\alpha(s^2v_1^{-1})\alpha^{-1})s^{-2}(v_1v_2v_3)^{-1} = st,$$

and (1.12) follows.  

**Remark.** We will exploit this fusion further in Section 3 when we describe the necessary and sufficient conditions that an element in $H^\bullet(\text{Syl}_2(O'N); F_2)$ be in $H^\bullet(O'N; F_2)$. 


2. THE LATTICE OF ELEMENTARY ABELIAN
SUBGROUPS IN G = O'N

In this section we describe the lattice of elementary abelian 2-subgroups
in \( G = \text{O'N} \). Our main sources are the Atlas [Co, ON, Yo] and a private
communication from Lyons [Ly]. To begin, we have

**Theorem 2.1.** There are exactly five distinct conjugacy classes of elementary abelian 2-groups in \( G = \text{O'N} \) described as follows:

<table>
<thead>
<tr>
<th>Group</th>
<th>Normalizer</th>
<th>Order of normalizer</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z \cong 2 )</td>
<td>( 4 \cdot L_1(4) \cdot 2 )</td>
<td>161,280 = ( 2^9 \cdot 3^2 \cdot 5 \cdot 7 )</td>
</tr>
<tr>
<td>( V_1 \cong 2^2 )</td>
<td>( 4^3 \cdot \sigma_4 )</td>
<td>1536 = ( 2^9 \cdot 3 )</td>
</tr>
<tr>
<td>( V_2 \cong 2^2 )</td>
<td>( (3^2 \cdot 4 \times \sigma_4) \cdot 2 )</td>
<td>864 = ( 2^9 \cdot 3^2 )</td>
</tr>
<tr>
<td>( E_1 \cong 2^3 )</td>
<td>( 4^3 \cdot L_1(2) )</td>
<td>10,752 = ( 2^9 \cdot 3 \cdot 7 )</td>
</tr>
<tr>
<td>( E_2 \cong 2^3 )</td>
<td>( (4 \times 2^3) \cdot \sigma_4 )</td>
<td>384 = ( 2^9 \cdot 3 )</td>
</tr>
</tbody>
</table>

**Proof.** The fact that there is only one conjugacy class of involutions in
\( G \) with centralizer \( 4 \cdot L_1(4) \cdot 2 \) appears in O'Nan's original paper [ON].
The presence of exactly two distinct conjugacy classes of Klein groups is
Lemma 3.1 in [Yo]; the normalizers of \( V_1, V_2 \) follow directly from the
discussion there.

Finally, in Lemma 3.2 in [Yo] it is proved that there are at most three
distinct conjugacy classes of \( 2^3 \) in \( G \), and that the 2-rank of \( G \) is three.
The representatives for these subgroups can be taken to be (using the
notation from Section 1):

\[
I' = \langle v_1^2, v_2^2, v_3^2 \rangle, \quad V' = \langle v_1^2 v_2^2, v_1^2 v_3^2, s^2 v_1^{-1} \rangle, \\
IV' = \langle v_2^2 v_3^2, v_1^2 v_2^2, st \rangle.
\]

(Note that in [Yo] it is shown that \( I' \) and \( V' \) are not conjugate.) However,
from our analysis of 2-fusion, (1.14), it follows that, in fact, \( IV' \) is conjugate
to \( V' \). The structure of the normalizers appears in [ON] as well as [Yo].

From this information we will now construct a simplicial complex
associated to this lattice. We recall its definition: consider the partially
ordered set of 2-elementary abelian subgroups in a finite group \( G \),
denoted by \( \mathcal{A}(G) \). Denote by \( |\mathcal{A}_2(G)| \) the simplicial complex associated to it;
it vertices are the elements of \( \mathcal{A}_2(G) \) and its simplices are the non-empty
finite chains (under inclusion). These complexes were first introduced by
Brown [B] and then studied by Quillen [Q]. They have a \( G \)-action induced
by conjugation.
Remark 2.2. In previous cases, such as the analysis of $M_{12}$ in [AMM] and the analysis of $G_2(q), 3D_4(q)$ in [FM], these poset spaces have allowed us to reduce the determination of $H^*(G; \mathbb{F}_2)$ to the analysis of a fairly small number of subgroups and their common intersections. As we will see in Section 5, this is also the case here. However, it turns out that in this case it is at least as hard to determine the cohomology of the subgroups as it is to study $H^*(O'N; \mathbb{F}_2)$ directly. This is similar to what occurred in our study of $M_{22}$ [AM2]. But, regardless, as we see in Section 4, the structure of $|\mathcal{F}(O'N)|/O'N$ will give us an important connection between the classifying space for $O'N$ and a union of two classifying spaces over a common intersection which holds out the hope of understanding some of the implications of the existence of $O'N$ in homotopy theory.

We now describe $|\mathcal{F}(O'N)|/O'N$.

Theorem 2.3. $|\mathcal{F}(O'N)|$ is a two-dimensional complex which, up to conjugacy (the action of $O'N$), has five vertices, eight 1-cells, and four 2-cells. The following is a list of conjugacy classes of flags with their normalizers:

<table>
<thead>
<tr>
<th>Flag</th>
<th>Normalizer</th>
<th>Order of normalizer</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(Z, V_1)$</td>
<td>$4^1 D_4$</td>
<td>512 = $2^9$</td>
</tr>
<tr>
<td>$(Z, V_1)$</td>
<td>$(3^1 \cdot 4 \times V_2)2$</td>
<td>288 = $2^5 \cdot 3^2$</td>
</tr>
<tr>
<td>$(Z, E_1)$</td>
<td>$4^1 \cdot \gamma_4$</td>
<td>1536 = $2^4 \cdot 3$</td>
</tr>
<tr>
<td>$(Z, E_2)$</td>
<td>$(4 \times 2^3) \cdot \gamma_4$</td>
<td>384 = $2^7 \cdot 3$</td>
</tr>
<tr>
<td>$(Z, E_3)$</td>
<td>$(4 \times 2^3)2^2$</td>
<td>64 = $2^6$</td>
</tr>
<tr>
<td>$(V_1, E_1)$</td>
<td>$4^1 \cdot \gamma_4$</td>
<td>1536 = $2^4 \cdot 3$</td>
</tr>
<tr>
<td>$(V_1, E_2)$</td>
<td>$(4 \times 2^3) \cdot D_8$</td>
<td>128 = $2^7$</td>
</tr>
<tr>
<td>$(V_2, E_1)$</td>
<td>$(4 \times V_2) \cdot \gamma_4$</td>
<td>96 = $2^3 \cdot 3$</td>
</tr>
<tr>
<td>$(Z, V_1, E_3)$</td>
<td>$4^1 \cdot D_8$</td>
<td>512 = $2^9$</td>
</tr>
<tr>
<td>$(Z, V_1, E_2)$</td>
<td>$(4 \times 2^3) \cdot D_8$</td>
<td>128 = $2^7$</td>
</tr>
<tr>
<td>$(Z, V_2, E_1)$</td>
<td>$(4 \times 2^3) \cdot 2^2$</td>
<td>64 = $2^6$</td>
</tr>
<tr>
<td>$(Z, V_2, E_2)$</td>
<td>$(4 \times V_2)2$</td>
<td>32 = $2^5$</td>
</tr>
</tbody>
</table>

Proof. The first four on the list are clear. The fifth one occurs because $N_2(E_3)$ does not act transitively on the seven subgroups of order 2 in $E_3$; there is an orbit with six elements and a fixed point. In [Yo] it is proved that no conjugate of $V_2$ lies in $E_1$, explaining the absence of $(V_2, E_1)$ and $(Z, V_2, E_1)$. The rest are then clear from these arguments.

$|\mathcal{F}(O'N)|$ is a two-dimensional CW complex with 2,079,117,579 vertices, 19,900,669,640 edges, and 26,100,878,270 faces. Hence $\chi(\mathcal{F}(O'N)) = 8,279,326,209 \equiv 1 \mod(2^9)$. The pictorial representation of the orbit space $|\mathcal{F}(O'N)|/O'N$ can be found in the Introduction.
3. THE DETAILED FUSION STRUCTURE FOR THE MAXIMAL ELEMENTARY SUBGROUPS

Each of the seven centralizers of maximal elements of (1.6) except for VI and VII is in a known conjugacy class in O'N: it is either that of I or II. We now show that one of the remaining two classes is conjugate to a subgroup of I and the other is conjugate to II.

In [AM1] a homomorphism \( \phi: \langle v_1, v_2, v_3, t, s^2 \rangle \rightarrow \text{Syl}_2(L_2(4)) \) was given with kernel \( \langle t, v_3^{-1} t v_3^2 \rangle \). This homomorphism cannot extend to an injection \( H_\beta \twoheadrightarrow 4 \cdot L_2(4); 2_1 \), so we begin our analysis by giving a homomorphism \( \phi: K_\beta \rightarrow \text{Syl}_2(L_3(4)) \), with kernel \( \langle v_1^{-1} v_2^3 \rangle \) which does extend to a homomorphism from \( H_\beta \) to \( L_3(4); 2_1 \), the group \( L_3(4) \) extended by the "unitary involution" (see below). Set \( \beta' = v_2 \cdot \beta \). \( \beta' \) is again of order three. We begin the construction by setting

\[
\phi(\beta') = \begin{pmatrix}
\zeta & 0 & 0 \\
0 & \zeta^2 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]

where \( \zeta \in \mathbb{F}_4 \) is a primitive third root of unity and we are in the quotient by the center \( L_3(4) \), and not in \( SL_3(4) \). We have \( H_\beta = K_\beta; 2_1 = \langle s^2, t, v_1, v_2, v_3 \rangle; \mathcal{S}_3 \), and a convenient choice for the extending \( \mathcal{S}_3 \) is

\( \mathcal{S}_3 = \langle v_1^{-1} v_2 t s^3, \beta' \rangle \),

which makes \( H_\beta \) explicit. Now remember that the element \( v_1^{-1} v_2 t s^3 \) is supposed to act on \( L_3(4) \) via the unitary involution \( A \mapsto M(A^{-1})M^{-1} \) where \( M \) is the matrix

\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix},
\]

so that when we specify the image of \( t \), the images of all the remaining generators in \( \langle v_1, v_2, v_3, t, s^2, \beta' \rangle \) are determined. Set

\[
\phi(t) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & \zeta \\
0 & 0 & 1
\end{pmatrix},
\]
so we obtain

\[
\begin{align*}
v_3^2 & \mapsto \begin{pmatrix} 1 & 0 & \xi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & v_1^2 & \mapsto \begin{pmatrix} 1 & 0 & \xi^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
t & \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \xi \\ 0 & 0 & 1 \end{pmatrix}, & ts^2v_1^{-1} & \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \\
ts \mapsto v_3^2v_1^{-1} & \mapsto \begin{pmatrix} 1 & \xi^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & v_1v_3^2v_1^{-1} & \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{align*}
\]

Remark. The choice for \( \beta' \) was made because it was required that \( \beta' \) should normalize \( \langle v_1v_3^{-1}v_2^2, t, s^2v_1^{-1} \rangle \), the group VII associated to (1.6), so the image of this group in \( L_3(4) \) is the subgroup of matrices of the form

\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}.
\]

A second, inequivalent choice is given by choosing \( \beta' \) to normalize the other group, \( VI = \langle v_1v_3^{-1}v_2^2, t, s^2v_1^{-1} \rangle \), and modifying \( ts^3 \) to an element \( xts^3 \) with \( x \in \langle v_1, v_2, v_3 \rangle \), so \( \langle xts^3, \beta' \rangle = \mathscr{N}_3 \).

Anyway, with the embedding in (3.1), the image of \( \langle v_1v_3^{-1}v_2^2, t, s^2v_1^{-1} \rangle \) is conjugate in \( 4 \cdot L_3(4) \) to the group

\[
\phi^{-1}\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \langle v_1v_3^{-1}v_2^2, v_3^2, v_1^2 \rangle.
\]

On the other hand, when we explicitly embed the remaining group \( \langle v_1v_3^{-1}v_2^2, v_4v_3t, s^2v_1^{-1} \rangle \) we find

\[
v_1v_3t \mapsto \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \xi \\ 0 & 0 & 1 \end{pmatrix}, & v_1v_3ts^2v_1^{-1} & \mapsto \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},
\]

which has determinant \( \det\begin{pmatrix} 1 & 1 \\ \xi & 1 \end{pmatrix} = \xi^2 \).
In particular, both groups are contained in the subgroup

\[ 4 \ast D_{4} \ast D_{4} = \langle v_{1}v_{3}^{-1}v_{1}^{2}, v_{3}^{2}, t, ts^{2}v_{1}^{-1} \rangle \sim \begin{pmatrix} 1 & 0 & e \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix} \subseteq 4 \cdot L_{3}(4), \]

which has the Weyl group \( \mathcal{W} \). Since this action fixes \( v_{1}v_{3}^{-1}v_{1}^{2} \), it is determined on the centralizers of the maximal elementary by what it does on their images in the quotient \( 2^{4} \subseteq L_{3}(4) \), and each such image is a \( 2^{2} \). It was pointed out in [AM1] that there are 4 conjugacy classes of \( 2^{2} \subseteq 2^{4} \equiv \mathbb{F}_{4}^{2} \) under this action of \( \mathcal{W} \), and they are distinguished by the determinant of any two non-zero elements in \( 2^{2} \). Thus, \( \langle v_{1}^{2}, v_{3}^{2}, v_{1}v_{3}^{-1} \rangle \) has determinant 0 as does \( \langle v_{1}v_{3}^{-1}v_{1}^{2}, t, s^{2}v_{1}^{-1} \rangle \), and they are conjugate. On the other hand, the determinant of \( \langle v_{1}v_{3}^{-1}, v_{3}^{2}, t \rangle \) is also \( \xi_{2} \) so this group is conjugate to \( \langle v_{1}v_{3}^{-1}, v_{3}^{2}, t, s^{2}v_{1}^{-1} \rangle \) as asserted.

This completes our analysis of the fusion in \( O'N \). There are five subgroups \( 2^{2} \) in one class, with centralizers all equal to \( 4 \times 2^{2} \) while there are two in the other class.

It remains to determine the explicit identifications of the six \( 4 \times 2^{2} \)'s, modulo the actions of their normalizers in order to make cohomology calculations.

We note that \( (4 \ast D_{4} \ast D_{4}); \mathcal{W} \) contains the normalizers in \( O'N \) of the maximal \( 2 \)-elementaries in II, III, V, and VI. The Weyl group \( \mathcal{W} \) of each centralizes \( v_{1}v_{3}^{-1}v_{1}^{2} \) and is generated by elements \( d, e, f, g \) with \( \langle d, e \rangle = 2^{2}, \langle e, f \rangle = \mathcal{W} \) inducing on \( \Pi' = \langle a^{2}, b, c \rangle \) the following actions—where we are writing \( a = v_{1}v_{3}^{-1}v_{1}^{2}, b = v_{3}^{2}, c = t; d: a^{2} \rightarrow a^{2}, b \rightarrow ba^{2}, c \rightarrow c; e: a^{2} \rightarrow a^{2}, b \rightarrow b, c \rightarrow a^{2}c; f: a^{2} \rightarrow a^{2}, b \rightarrow b, c \rightarrow b; g: a^{2} \rightarrow a^{2}, b \rightarrow c, c \rightarrow bc \). (Since the normalizer of \( \Pi' \) modulo \( \Pi \) acts faithfully on the \( 2^{2} \)-subgroup \( \Pi' \) fixing \( v_{1}v_{3}^{2} \), it can be identified with a maximal parabolic subgroup \( 2^{2}: L_{3}(2) \equiv \mathcal{W} \subseteq L_{3}(2) \) and the claim follows.)

Thus the action is uniquely determined on II, III, V, and VI. Moreover, the fusions of these groups in \( 4 \cdot L_{3}(4) \); \( 2_{1} \) are also clear, since \( v_{1}v_{3}^{-1}v_{1}^{2} \) is fixed by every element in \( 4 \ast D_{4} \ast D_{4}; \mathcal{W} \). It remains to determine explicitly the fusion between IV and V. We have, referring to the calculations in the proof of Lemma 1.12,

\[
\begin{align*}
\alpha(v_{1}v_{3}^{-1}v_{1}^{2}) &= v_{1}^{2}(v_{2}v_{3})^{-1}, \\
\alpha((v_{1}v_{3}^{2})^{2}) &= (v_{1}v_{3}^{2})^{2}, \\
\alpha(s^{2}v_{1}^{-1}) &= v_{1}^{2}v_{3}ts^{3},
\end{align*}
\]
and conjugating by $v_1v_2v_3s^2$ gives the explicit map

$$v_1v_2v_3^{-1}v_2 \mapsto v_1^2v_2v_3,$$

$$(v_1v_2)^2 \mapsto (v_1v_3)^2,$$

$$s^2v_3^{-1} \mapsto st.$$

Thus, the subgroup $4$ fixed under the action of the normalizer of $IV$ is $\langle v_2v_3^{-1}(v_1v_2)^2 \rangle$.

Finally, VII is identified with $\langle v_1v_2v_3^{-1}v_2^2, v_1^2, v_2^2 \rangle$ in $4 \ast D_8 \ast D_4$ by an element which fixes $v_1v_2^{-1}v_2^3$, and this completes the explicit descriptions of the fusions.

Now we recall one of the results of [M2] where the cohomology ring of $\text{Syl}_2(O'N)$ is determined.

**Theorem 3.2.** $H^*(\text{Syl}_2(O'N); \mathbb{F}_2)$ is detected by restriction to the seven centralizers of the seven distinct conjugacy classes of elementary two subgroups.

Consequently, we have

**Corollary 3.3.** Identify the five subgroups II, III, IV, V, VI and identify VII with $\langle v_1v_2v_3^{-1}v_2^2, v_1^2, v_2^2 \rangle$ as discussed above. Then, $v \in H^*(\text{Syl}_2(O'N); \mathbb{F}_2)$ is in the restriction image from $H^*(O'N; \mathbb{F}_2)$ if and only if its restrictions to these seven centralizers are all the same under these identifications and they belong to the invariant subrings in cohomology.

**Remark 3.4.** Of course the same techniques may be applied to the groups $\text{Alp}_2$ and $4 \cdot L_2(4)$. For example, with $\text{Alp}_2$ the fusions are as follows: $\beta$ and $\alpha$ fuse II, III, IV, and V; but the remaining groups I, VI, and VII are not fused. Here the Weyl groups are $D_8$ for II, and $L_2(2), \mathcal{A}_4$, and $\mathcal{A}_4$ for I, VI, and VII respectively.

Thus we get

**Corollary 3.5.** $a \in H^*(\text{Syl}_2(O'N); \mathbb{F}_2)$ is in the image from $H^*(\text{Alp}_2; \mathbb{F}_2)$ if and only if

1. $\text{Res}(a) \in H^*(I; \mathbb{F}_2)^I_{\text{ad}_1},$
2. $\text{Res}^I(a) = \text{Res}^{III}(a) = \text{Res}^{IV}(a) = \text{Res}^{V}(a),$
3. $\text{Res}^{VI}(a) \in H^*(VI; \mathbb{F}_2)^{VI}_{\mathcal{A}_4},$
4. $\text{Res}^{VI}(a) \in H^*(VII; \mathbb{F}_2)^{VI}_{\mathcal{A}_4}.$

**Remark 3.6.** The situation for $4 \cdot L_2(4)$. $2_1$ is that $I$ is fused with VII, while II, III, V, and VI fuse with Weyl group $\mathcal{A}_4$. But IV is not fused with any other subgroup. The Weyl group of $I$ is also $\mathcal{A}_4$. 
Further discussion of the actual cohomology of $O'N$ is deferred to [M2]. However, we now record the generators of $H^*(\text{Syl}_2(O'N); \mathbb{F}_2)$ and their restrictions to the seven conjugacy classes above from [M2].

First, we have three polynomial generators $d_8$, $d_{12}$, and $d_{14}$, which come from the Stiefel–Whitney classes of a representation of $O'N$. (Note that the first is indecomposable in $H^*(\text{Syl}_2(O'N); \mathbb{F}_2)$ but the second and third are sums of decomposables.)

The image of $H^*(E_1; \mathbb{F}_2)$ in $H^*(E_1; \mathbb{F}_2)$ is $\mathbb{F}_2[x_i(1)^2, x_i(2)^2, x_i(3)^2]$, and clearly

$$\text{im}(\text{res}^*: H^*(O'N; \mathbb{F}_2) \to H^*(E_1; \mathbb{F}_2)) \subseteq \text{im}(\text{res}^*: H^*(4^3 \cdot L_3(2); \mathbb{F}_2)) \to H^*(E_1; \mathbb{F}_2)) \subseteq \mathbb{F}_2[x_i(1)^2, x_i(2)^2, x_i(3)^2]^{L_3(2)}.$$ 

By a well known result of Dickson,

$$\mathbb{F}_2[x_i(1)^2, x_i(2)^2, x_i(3)^2]^{L_3(2)} = \mathbb{F}_2[d_4^2, d_6^2, d_8^2],$$

where

$$d_4 = x_i(1)^4 + x_i(2)^4 + x_i(3)^4 + x_i(1)^2 x_i(2) x_i(3) + x_i(1) x_i(2)^2 x_i(3) + x_i(1) x_i(2) x_i(3)^2,$$

$$d_6 = Sq^2(d_4), \quad d_8 = Sq^4(d_4),$$

and we have

**Theorem 3.7.** The image under restriction of $H^*(O'N; \mathbb{F}_2)$ in $H^*(E_1; \mathbb{F}_2)$ is exactly the algebra $\mathbb{F}_2[d_4^2, d_6^2, d_8^2]$. Moreover, the image under restriction in $H^*(E_2; \mathbb{F}_2)$ contains a copy of this same algebra.

**Proof.** To obtain this result, we use the Stiefel–Whitney classes of a representation. We recall a few basic facts. If $O(n)$ is the $n \times n$ orthogonal group over the real numbers, and $\rho: G \to O(n)$ is a group homomorphism, then the Stiefel–Whitney classes of $\rho$, denoted $w_i(\rho) \in H^i(BO(n), \mathbb{F}_2)$, are by definition $\rho^*(w_i)$, where $w_1, w_2, \ldots, w_n$ are the well-known symmetric polynomial generators for the cohomology of $BO(n)$. The total Stiefel–Whitney class is the formal sum $W(\rho) = w_1(\rho) + \cdots + w_n(\rho)$. Note that $W(\rho_1 \oplus \rho_2) = W(\rho_1)W(\rho_2)$. We refer the reader to [MS] and [AM] for more on this.

In [Co] the irreducible representations of $O'N$ and their characters are given. We concentrate on $\chi_{16}$, which has dimension 64790. Any element of order 2 in $O'N$ has character 70 under this representation. Moreover, the representation is real. Hence, when restricted to $E_1$ or $E_2$, the representa-
tion has the form $70e + (8090)R$ where $e$ is the trivial representation and $R$ is the regular representation. The total Stiefel–Whitney class of $R$ is $1 + d_4 + d_6 + d_7$, and consequently the total Stiefel–Whitney class of $\chi_{16}$, when restricted to $E_1$ or $E_2$, is

$$(1 + d_4 + d_6 + d_7)^{2^{r+8} + \cdots} = 1 + d_4^2 + d_6^2 + d_7^2 + \cdots.$$ 

In particular, in both $H^*(E_1; \mathbb{F}_2)$ and $H^*(E_2; \mathbb{F}_2)$ the classes $d_4^2$, $d_6^2$, and $d_7^2$ are in the image of restriction from $H^*(O'N; \mathbb{F}_2)$. Equation (3.7) follows. 

We have included Table I describing the restrictions (on the generators) from the cohomology of the 2-Sylow subgroup of $O'N$ to the cohomology of the centralizers of the seven distinct conjugacy classes of elementary abelian 2-subgroups.

To illustrate the techniques, note that $e_7 e_8 + e_7 \Gamma(x)_2 + L_3 + S_7 + V_3$ restricts to 0 at I, VII, while it restricts to $d_3$ at each of II–VI. Consequently, it is in the image of $H^*(O'N; \mathbb{F}_2)$. Also, $M_4$ is directly seen from the table to lie in the image of $H^*(O'N; \mathbb{F}_2)$. Since $Sq^3(d_4) = d_2 d_3$, this is also an element in the image, representing $Y_6$. It thus remains only to construct $X_3$. But this is represented by

$$d_3(e_7^2 + (e_7^2 + e_7^2 + e_7^4) + e_7^4 + e_7^2 + \alpha_8^2) + S_7 + L_3 \Gamma(w)_4 + \Gamma(x)_4 S_7.$$

It follows that $H^*(O'N; \mathbb{F}_2)$ is at least as large as asserted in the Introduction. To see that it can be no larger requires a close look at the possible invariant classes in the restriction image in $H^*(IV; \mathbb{F}_2)$. The only possibility which is not already in the image from $H^*(O'N; \mathbb{F}_2)$ is $\lambda d_3^2 d_3$, but to build this class requires $\Gamma(w)$, and by looking simultaneously in $H^*(V; \mathbb{F}_2)$ we see that this class cannot occur in the restriction image from $H^*(O'N; \mathbb{F}_2)$. Further details will be found in [M2].

Remark. In a similar way we can construct classes in the image of $H^*(\text{Alp}_3^2; \mathbb{F}_2)$. For example, $L_3 + e_7 \Gamma(x)$ is in the image, as is $e_7^2 + e_7 e_7 + e_7^2$.

4. APPLICATIONS OF A FORMULA OF WEBB

We will now apply a formula due to Webb [We] to the poset complex of $O'N$ discussed in Section 2. Let $G$ be any finite group, $\Gamma(G)$ be the poset complex as above, and denote the rank of $G$ at 2 by $r(G)$. Then Webb’s result can be restated as follows (see [AM, Chap. V] for details). Let $C'$
<table>
<thead>
<tr>
<th>Name: Group</th>
<th>I: $\langle r_1, r_2, r_3 \rangle$</th>
<th>II: $\langle t, r_2^2 \rangle$</th>
<th>III: $\langle t, r_2, r_3 \rangle$</th>
<th>IV: $\langle l, (r_1 r_2)^3, s, t \rangle$</th>
<th>V: $\langle j, r_2^2, s, t \rangle$</th>
<th>VI: $\langle j, t, s^2 r_1^{-1} \rangle$</th>
<th>VII: $\langle j, t, s^2 t r_1^{-1} \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_i$</td>
<td>0</td>
<td>$h$</td>
<td>$h$</td>
<td>$h$</td>
<td>$h$</td>
<td>$h$</td>
<td>$h$</td>
</tr>
<tr>
<td>$e_i$</td>
<td>$e(1) + e(2) + e(3)$</td>
<td>$0$</td>
<td>$h$</td>
<td>$h$</td>
<td>$h$</td>
<td>$h$</td>
<td>$h$</td>
</tr>
<tr>
<td>$e_i$</td>
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<td>$0$</td>
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<td>$0$</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>$b(1) + b(2) + b(3)$</td>
<td>$S + \lambda h$</td>
<td>$S + \lambda h$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\lambda h$</td>
<td>$\lambda h$</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>$\lambda_3$</td>
<td>$\lambda_2$</td>
<td>$\lambda_2$</td>
<td>$d_3 + h^2 \lambda$</td>
<td>$0$</td>
<td>$\lambda h(b + h)$</td>
<td>$\lambda h(b + h)$</td>
</tr>
<tr>
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<td>$\lambda_2$</td>
<td>$\lambda_2$</td>
<td>$\lambda_2$</td>
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<td>$0$</td>
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<td>$\lambda_2$</td>
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<td>$\lambda h$</td>
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<td>$\lambda h$</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>$\lambda_2$</td>
<td>$\lambda_2$</td>
<td>$\lambda_2$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\lambda d_2 + d_3$</td>
<td>$\lambda d_2$</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>$\lambda_3$</td>
<td>$\lambda_2$</td>
<td>$\lambda_2$</td>
<td>$\lambda h^2$</td>
<td>$\lambda h^2$</td>
<td>$\lambda h^2$</td>
<td>$\lambda h^2$</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>$\lambda_2$</td>
<td>$\lambda_2$</td>
<td>$\lambda_2$</td>
<td>$\lambda h^2$</td>
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<td>$\lambda_2$</td>
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<td>$\lambda h^2$</td>
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<td>$\lambda_3$</td>
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<td>$\lambda h^2$</td>
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</tbody>
</table>

Note. We have not included $d_6$, $d_{12}$, and $\tilde{d}_{12}$ in this table. Their restrictions have already been described in (3.7). As regards the elements in the table we have used the following abbreviations: $\mathcal{A} = (b(1) + b(3) e(2) + e(1) + e(3)) b(2), S = b(h + h), N_1 = (d_3 + h^2 \lambda)^2, \mathcal{F} = (e(1) + e(2) + e(3)) b(2) k(1) + b(3), \lambda = (b(1) + b(2) + b(3)) M_4 + S \theta^2 M_4 - b(3) k(1) + b(2) e(1) e(2) + b(2) k b(1) + b(3) e(1) e(2) + b(1) k^2 b(2) + b(3) k e(2) e(3).$ For the groups II–VII, the element $b$ is dual to the middle generator, e.g., $t$ in II, $r_2^3$ in III; $h$ is dual to the third generator, e.g., $r_2^3 t$ in III; $\lambda$ is dual to the first generator, and $b(h) = \beta_2 \lambda).$ Also, $j = \gamma r_1^{-2}, l = \gamma r_1^{-2} (r_1 r_3), t = \gamma r_1^{-2} (r_1 r_3)^2.$ This table is set up so that the fusion of the various subgroups preserves the form of expressions $b \rightarrow b, h \rightarrow h, \lambda \rightarrow \lambda.$ Finally, as usual, $d_2 = h^2 + bh + b^2, d_3 = b^2 h + bh^2.$
denote the $i$th cellular chain group over $\mathbb{F}_2$ of the $G$-CW complex $|\mathcal{A}_2(G)|$; then the coboundary map defines a cochain complex $\mathcal{D}^*(q) = H^*(G, C^*)$:

$$H^q(G, C^0) \to H^q(G, C^1) \to \cdots \to H^q(G, C^{n(G)-1})$$

with $H^r(\mathcal{D}(q)) = 0$ if $r > 0$ and $H^0(\mathcal{D}(q)) \cong H^0(G)$.

In particular, taking Euler characteristics gives the formula

$$H^* (G; \mathbb{F}_2) \oplus \bigsqcup_{\sigma \in [\mathcal{A}_2(G)]/G} H^* (G_{\sigma}; \mathbb{F}_2) \cong \bigsqcup_{\sigma \in [\mathcal{A}_2(G)]/G, i \text{ even}} H^* (G_{\sigma}; \mathbb{F}_2)$$

where $\sigma_i$ is a representative for an orbit of $i$-cells in $|\mathcal{A}_2(G)|$. For the group $O'N$ this formula simplifies drastically after cancellation and we have the following

**Theorem 4.1.** There is an isomorphism

$$H^* (O'N; \mathbb{F}_2) \oplus H^* (4^4 \cdot \mathcal{A}_4; \mathbb{F}_2) \cong H^* (4 \cdot L_3(4); \mathbb{F}_2)$$

$$\oplus H^* (4 \cdot L_3(2); \mathbb{F}_2).$$

We can draw several conclusions from the structure of the poset space and its quotient.

**Theorem 4.2.** The 2-local structure of $O'N$ defines a surjective homomorphism, $\pi$, from an amalgamated product to $O'N$,

$$\pi: 4 \cdot L_3(4); 2_1 \ast_4 4^3 \cdot L_3(2) \to O'N,$$

and $\pi$ induces isomorphisms in mod(2) homology and cohomology.

**Proof.** By a theorem of Brown [B] and Webb’s formula we have an isomorphism $H^* (|\mathcal{A}_2(O'N)| \times_{O'N} E_{O'N}; \mathbb{F}_2) \cong H^* (O'N; \mathbb{F}_2)$ where $E_{O'N}$ is the universal cover of the classifying space of $O'N$. Consider the projection

$$q: |\mathcal{A}_2(O'N)| \times_{O'N} E_{O'N} \to |\mathcal{A}_2(O'N)|/O'N.$$

The inverse image of every open simplex in $|\mathcal{A}_2(O'N)|/O'N$ is the product of that open simplex with the classifying space of the isotropy group of that simplex, and this gives a decomposition of a space having the same homology as $O'N$. Now, consider the leftmost triangle.

```
4 \cdot L_3(4); 2_1
(3^4 \cdot A_4); 2
(4^3 \cdot A_4); 2
(4 \times 2^2); 2
(4 \times 2^2); S_4
(4 \times 2^2); S_4
```


Note that the top edge and the entire simplex have isotropy groups with the same mod(2) cohomology. Hence, they cancel out geometrically as well as algebraically, and we can remove the edge and the simplex. Note also that the leftmost vertex and the lower edge also have isotropy groups with the same mod(2) cohomology, and so we are left with only the right edge. A similar consideration holds for the curved edge with isotropy group \((4 \times 2^2)S_i\) which is exactly the same isotropy group as that of the "curved" triangle, so the two cancel out. We may now cancel the lower edge of the central triangle (with isotropy group \((4 \times 2^2)D_s\)) and the interior of the triangle. That leaves us with the edge with isotropy group \((4 \times 2^2)S_i\) and the rightmost triangle with isotropy group \(Syl_3(O'N)\). This triangle cancels with its lower edge. The edge cancels with its lower vertex, and finally the lower right-hand edge cancels with the vertex with isotropy group \(4 L_3(2)\). This leaves only the diagram

\[
\begin{array}{c}
\bullet \\
4 \cdot L_3(4) : 2_1 \\
\bullet \\
4^3 \cdot L_3(2).
\end{array}
\]

Algebraically what we have is that the chain complex \(C^*(q)\) is homologous to the one-dimensional sub-cochain complex

\[
H^q(4 \cdot L_3(4) : 2_1) \oplus H^q(4^3 \cdot L_3(2)) \rightarrow H^q(4^3 \cdot S_i),
\]

hence from Webb’s theorem we conclude that the sum of the restriction maps is onto, and that its kernel is precisely \(H^i(O'N)\). On the other hand, if we denote by \(\Gamma\) the corresponding amalgamated product, the associated Mayer–Vietoris sequence also implies that the kernel is isomorphic to \(H^i(\Gamma)\). Consequently, by rank considerations and the fact that the finite subgroups in \(\Gamma\) are mapped isomorphically into \(O'N\) by the projection \(\pi\), we obtain the desired result. □

Identifying the kernel of the sum of the restrictions, we obtain

**Corollary 4.4.** \(H^*(O'N; \mathbb{F}_2)\) is given explicitly as the intersection in \(H^*(H_0; \mathbb{F}_2)\) of the images, \(\text{im}(H^*(4 \cdot L_3(4) : 2_1; \mathbb{F}_2))\) and \(\text{im}(H^*(4^3 \cdot L_3(2); \mathbb{F}_2))\),

\[
H^*(O'N; \mathbb{F}_2) = \text{im}(H^*(4 \cdot L_3(4) : 2_1; \mathbb{F}_2)) \cap \text{im}(H^*(4^3 \cdot L_3(2); \mathbb{F}_2)) \subset H^*(H_0; \mathbb{F}_2).
\]

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COHOMOLOGY OF O'NAN'S GROUP


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