INVARIANTS AND COHOMOLOGY OF GROUPS
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This paper is dedicated to the memory of José Adem (1921-1991)

1. Introduction

Let $G$ be a finite group and $p$ a prime number dividing its order. The effective calculation of its cohomology ring $H^*(G; \mathbb{F}_p)$ becomes difficult as soon as the $p$-rank of $G$ is larger than one. There are, however, well-known local methods ([Q1], [W1]) which allow the use of the cohomology rings of the normalizers of elementary abelian $p$-subgroups of $G$ to determine the cohomology of $G$.

These methods are quite effective, reducing the question to studying the cohomology rings and cohomology maps induced from a lattice of proper subgroups of $G$ (usually each much smaller than $G$). However, the methods run into difficulties as soon as $G$ normalizes one of the elementary $p$-subgroups. Thus, from a calculational point of view, extensions of the form

$$1 \rightarrow V \rightarrow G \rightarrow K \rightarrow 1$$

with $V \cong (\mathbb{Z}/p)^n$ are of great importance in group cohomology. In particular we have an induced restriction map

$$H^*(G; \mathbb{F}_p) \rightarrow H^*(V; \mathbb{F}_p)$$

indicating that rings of invariants play an important part in many calculations. Indeed, existing results for the symmetric groups [AMM1] and the general linear groups [Q2] rely heavily on determining rings of invariants.

In this paper we will describe a cohomological decomposition for group extensions where rings of invariants play a significant role. To state it we need to recall a few definitions. For a finite group $K$ let $|A_p(K)|$ denote the geometric realization of the partially ordered set of elementary abelian $p$-subgroups of $K$. For any $i$-simplex $\sigma_i$ in this $K$-complex, we denote its stabilizer by $K_{\sigma_i}$ and its orbit representative by $[\sigma_i]$. We then have

**Theorem (2.2).** Let

$$1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$$

be an extension of finite groups. Then, with $\mathbb{F}_p$ coefficients we have an isomorphism

$$H^*(G; \mathbb{F}_p) \oplus \left( \bigoplus_{[\sigma_i], \text{odd}} H^*(\pi^{-1}(K_{\sigma_i})) \right) \oplus \left( \bigoplus_{[\sigma_i], \text{even}} H^*(H)^{K_{\sigma_i}} \right)$$

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where the \([\sigma_i]\) run over the simplexes of \(|A_p(K)|/K\).

The point of this expression is that it measures very precisely how far \(H^*(G)\) differs from \(H^*(H^K)\); note in particular the extreme case \((|K|, p) = 1\) when \(H^*(G) \cong H^*(H^K)\). The proof of this theorem relies on a splitting for group cohomology due to P. Webb \([W1]\) together with some techniques from equivariant cohomology.

As particular examples we note that there are a number of extensions of the form

\[(\mathbb{Z}/2)^n \rightarrow G \rightarrow A_m\]

where \(n = 3\) or \(4\) and \(A_m\) is the alternating group on \(m\) letters which are critical in studying the cohomology of some of the sporadic groups. More precisely, there are representations

\[A_n \rightarrow \text{GL}_4(\mathbb{F}_2) \cong A_8\]

for \(n = 5, 6, 7\) (two distinct representations of \(A_5\)) which give rise to rings of invariants involved in the cohomology of the sporadic simple groups \(O'N\), \(M_{22}\), \(M_{23}\), and \(M^{10}\) which are critical ingredients in the determination of their cohomology rings \((\text{AM2}, \text{AM3})\). Many of these examples are discussed in §4.

After we have proved (2.2) in §2 we give a detailed computer assisted determination of the rings of invariants mentioned above. Some of the invariant subrings of \(\text{GL}_4(\mathbb{F}_2)\) are determined in \([AM1]\). Here we determine most of the remaining rings which play a role in the structure of the sporadic groups. The main result along these lines is

**Theorem.**

1. There are two non-conjugate copies of \(A_5 \subset A_8\). The ring of invariants for the first is given in \([AM1]\) and for the second its invariant subring in \(C = \mathbb{F}_2[x_1, \ldots, x_4] \cong \mathbb{F}_2[v_2, v_3, v_4, v_5](1, b_{10})\).

2. \(C^{A_6} \cong \mathbb{F}_2[\gamma_3, \gamma_5, D_8, D_{12}](1, \gamma_9, b_{15})\). Here, \(\gamma_3 = \text{Sq}^1(\sigma_2)\) where \(\sigma_2\) is the symmetric monomial in \(x_1, \ldots, x_4\), \(\gamma_5 = \text{Sq}^2(\gamma_3)\) and \(\gamma_9 = \text{Sq}^4(\gamma_5)\).

3. \(C^{A_7} \cong \mathbb{F}_2[D_8, D_{12}, D_{14}, D_{15}](1, d_{18}, d_{21}, d_{22}, d_{23}, d_{24}, d_{27}, d_{45})\).

This is a summary of the main results of §3. In all cases the subscript denotes the dimension of the generator and the \(D_i\) are the generators for the Dickson algebra \(C^{\text{GL}_4(\mathbb{F}_2)}\).
Remark. The ring in (1) above occurs in studying the Janko groups $J_2$ and $J_3$. In particular, in $J_2$, the normalizer of one of the two conjugacy classes of involutions has the form $2^{1+4} \rtimes A_5$, and the quotient semi-direct product $(\mathbb{Z}/2)^4 \rtimes A_5$, is given via this action. The other invariant subrings are needed in studying the cohomology of the Mathieu groups $M_{22}, M_{23}$, and the group $O'N$.

There is every indication that similar rings of invariants play an essential role in other cohomology calculations, hence it is important to develop systematic methods for computing them. In this regard Peter Kropholler informs us that he has general results on the rings $\mathbb{F}_2[x_1, \ldots, x_{2n}]^{Sp_{2n}(\mathbb{F}_2)}$ and $\mathbb{F}_2[x_1, \ldots, x_n]^{On(\mathbb{F}_2)}$. For $Sp_{2n}(\mathbb{F}_2)$, he shows that the ring of invariants has the form

$$\mathbb{F}_2[\gamma_3, \gamma_5, \gamma_9, \ldots, \gamma_{2n-2+1}, D_{2n-1}, \ldots, D_{2n-2}](1, \gamma_{2n-1+1})$$

which explains most of the elements in part (2) of our theorem.

In §4 we apply (2.2) to decompose the mod(2) cohomology of certain sporadic simple groups explicitly exhibiting the contribution of the invariants discussed above. In particular we give a complete discussion of the Mathieu group $M_{11}$, replacing the much longer work of [BC]. We also discuss some of the relations between the Mathieu group $M_{12}$ and the exceptional Chevalley groups $G_2$ and $G_2(q)$. Further details will appear in [AM2], [AM3]. Unless otherwise indicated, $\mathbb{F}_p$ coefficients will be assumed throughout.

This paper is dedicated to the memory of José Adem. He was an elegant mathematician whose understanding of classical questions in algebra led to far-reaching contributions in algebraic topology. As an individual he was an example of that rare combination of style and substance. To one of us he was a valued friend whose support when he was just starting out was crucially appreciated and whose ideas and comments were very very important in his early work. To the other he was a lifelong example and inspiration.

2. Invariants from local methods

We consider a fixed extension

$$1 \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow 1$$

and let $p$ be a prime dividing the order of $G$. The image of the restriction map

$$(i_H^G)_*: H^*(G) \longrightarrow H^*(H)$$

lies in the ring of invariants $H^*(H)^K$, but $i^*$ is neither injective nor surjective in general. However, if $(|K|, p) = 1$ then using the transfer map $tr^*: H^*(H) \longrightarrow H^*(G)$ together with the fact that $i^*tr^*: H^*(H) \longrightarrow H^*(H)$ is the
sum $\sum_{k \in K} k^*$ which is valid when $H$ is normal in $G$ we see that $\text{im}(i^*)$ is exactly $H^*(H)^K$, moreover, using the general fact that $\text{tr}^* i^*: H^*(G) \to H^*(G)$ is just multiplication by $|G:H|$ we have

$$H^*(G) \cong H^*(H)^K.$$ 

The goal of this section is to see how this situation changes when $\text{Syl}_p(K) \neq 1$ and to develop a systematic method for evaluating this discrepancy.

Consider the partially ordered set $A_p(K)$ of $p$-elementary abelian subgroups of $K$, and let $X = |A_p(K)|$ be the geometric realization of the associated category consisting of objects in $A_p(K)$ and maps (proper) inclusions. $X$ is a finite cell complex with a cellular $K$-action induced by conjugation. An $n$-simplex $\sigma \in X$ corresponds to a flag

$$F(\sigma) = \{0\} \neq (\mathbb{Z}/p)^i \subset \cdots \subset (\mathbb{Z}/p)^{i+1}.$$ 

Let $K_\sigma$ denote the stabilizer of $F(\sigma)$. Then the action of $K$ on $A_p(K)$ extends to a simplicial action on $X$ and the fixing subgroup of $K$ on the simplex $\sigma$ is exactly $K_\sigma$. We recall a theorem due to P. Webb [W1]

**Theorem (2.1).** There exist projective $\mathbb{F}_pK$-modules $P$ and $Q$ so that

$$\mathbb{F}_p \oplus \left( \bigoplus_{[\sigma_i], \text{i even}} \mathbb{F}_p[K/K_{\sigma_i}] \right) \oplus P \cong_K \left( \bigoplus_{[\sigma_i], \text{i odd}} \mathbb{F}_p[K/K_{\sigma_i}] \right) \oplus Q$$

where the $[\sigma_i]$ range over orbit representatives in $(X/K)^{(i)}$. 

The virtual projective module $P - Q$ is called the Steinberg module of $K$ at $p$ as it generalizes the usual projective module arising from a Tits building [W2].

Returning to our extension let $G_\sigma = \pi^{-1}(K_\sigma) \subset G$. It is reasonable to assume that the $G_\sigma$ should play a part in any cohomological splitting for $H^*(G)$ arising from $K$; however, the representation theoretic discrepancy in (2.1) (expressed as $P - Q$) gives rise to an "error term" involving invariants. The precise result is

**Theorem (2.2).** Let

$$1 \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow 1$$

be an extension of finite groups. Then, with $\mathbb{F}_p$ coefficients we have an isomorphism

$$H^*(G) \oplus \left( \bigoplus_{[\sigma_i], \text{i odd}} H^*(\pi^{-1}(K_{\sigma_i})) \right) \oplus \left( \bigoplus_{[\sigma_i], \text{i even}} H^*(H)^{K_{\sigma_i}} \right)$$
Proof. Let $EG$ denote a free, contractible $G$-CW complex with mod$(p)$ cellular cochain complex $C^*(EG)$. $H$ acts freely on $EG$ so $EG/H \cong BH$. Also $BH$, in this representation, inherits a free $K$-action realizing $BH$ as a principal $K$ cover of $BG = EG/G$. Tensor formula (2.1) with $C^*(EG/H)$. This yields a corresponding $K$-isomorphism of cochain complexes,

$$\cong H^*(H)^K \oplus \left( \bigoplus_{[\sigma], i \text{ even}} H^*(\pi^{-1}(K_{\sigma})) \right) \oplus \left( \bigoplus_{[\sigma], i \text{ odd}} H^*(H)^{K_{\sigma}} \right).$$

Apply the functor $H^*(K; -)$ to this equation where we regard the tensor product of two $\mathbb{F}_pK$-modules as a $\mathbb{F}_pK$-module via the usual rule $k(a \otimes b) = k(a) \otimes k(b)$ when $k \in K$. Note that for any subgroup $L \subseteq K$ we have

$$H^*(K; C^* \otimes \mathbb{F}_p[K/L]) \cong H^*(L; C^*/L).$$

On the other hand we also have the following result

**Lemma (2.3).** For any subgroup $L \subseteq K$, we have

$$H^*(L; C^*(EG/H)) \cong H^*(\pi^{-1}(L)).$$

**Proof of (2.3).** The left hand side computes the cohomology of

$$EL \times_L EG/H = (EL \times EG/H)/L \simeq (EG/H)/L \simeq B\pi^{-1}(L).$$

At this point the only terms which remain to be understood are $H^*(K; C^* (EG/H) \otimes P)$ and $H^*(K; C^*(EG/H) \otimes Q)$. For this step we need

**Lemma (2.4).** If $P$ is a projective $\mathbb{F}_pK$ module and $C^*$ is any $\mathbb{F}_pK$ cochain complex, then

$$H^*(K; C^* \otimes P) \cong [H^*(C) \otimes P]^K.$$

**Proof of (2.4).** There is a spectral sequence with

$$E_2^{pq} = H^p(K; H^q(C^* \otimes P)) \Rightarrow H^{p+q}(K; C^* \otimes P).$$
However, as $P$ is projective we have

$$H^p(K; H^q(C \otimes P)) \cong H^p(K; H^q(C) \otimes P) \cong \begin{cases} 0 & \text{for } p > 0, \\ (H^q(C) \otimes P)^K & \text{for } p = 0. \end{cases}$$

Thus, the spectral sequence collapses at $E_2$ and (2.4) follows.

We return to the proof of (2.2). Take (2.1), tensor with $H^*(H)$, and apply $K$-invariants to it. From the discussion above we obtain

$$H^*(H)^K \oplus \left( \bigoplus \limits_{\{\sigma_i\}, \text{ i odd}} H^*(H)^{K_{\sigma_i}} \right) \oplus (H^*(H) \otimes P)^K \cong \left( \bigoplus \limits_{\{\sigma_i\}, \text{ i even}} H^*(H)^{K_{\sigma_i}} \right) \oplus (H^*(H) \otimes Q)^K,$$

or, stated in virtual terms,

$$(H^*(H) \otimes St(K))^K \cong \left( \sum (-1)^i H^*(H)^{K_{\sigma_i}} \right) - H^*(H)^K.$$ 

When we substitute this expression (2.2) follows.

**Remark.** Instead of $|A_p(K)|$ we could have used any other $K$-complex for which (2.1) holds. These include $|S_p(K)|$ (the poset of non-trivial $p$-subgroups) or the Tits buildings if $K$ is of Chevalley type, and for $M_{22}$ the results of [RSY] give another lattice of subgroups distinct from these which also satisfy (2.1). See Example 4.4 for further details.

We now give several examples where (2.2) is useful, starting with familiar extensions. The following notation will be useful from here on: we set $V_n = (\mathbb{Z}/2)^n$ and we regard $V_n$ as the $n$-dimensional vector space over the field $\mathbb{F}_2$.

**Example (2.5).** The symmetric group on 4 letters, $\Sigma_4$, is given as the extension

$$1 \longrightarrow K (\cong \mathbb{Z}/2 \times \mathbb{Z}/2) \longrightarrow \Sigma_4 \longrightarrow SL_2(\mathbb{F}_2) (\cong \Sigma_3) \longrightarrow 1$$

where $K$ is the Klein group. The poset space $|A_2(\Sigma_3)|$ consists of three copies of $\mathbb{Z}/2$ so the orbit space consists of a single point with isotropy group $\mathbb{Z}/2$, and we have

$$H^*(\Sigma_4) \otimes H^*(V_2)^{\mathbb{Z}/2} \cong H^*(V_2)^{SL_2(\mathbb{F}_2)} \oplus H^*(D_8).$$

Now, $H^*(V_n)^{\mathbb{Z}/2} = \mathbb{F}_2[x_1, \ldots, x_n]^{\Sigma_n} = \mathbb{F}_2[w_1, w_2, \ldots, w_n]$, where $w_i$ denotes the $i^{th}$ symmetric power of the monomials $x_i$, the well known symmetric algebra, and

$$\mathbb{F}_2[x_1, \ldots, x_n]^{GL_n(\mathbb{F}_2)} = \mathbb{F}_2[D_{2n-1}, D_{2n-2}, \ldots, D_{2n-1}].$$
the well known Dickson algebra. Here the \(D_{2n-2i}\) are the Dickson invariants. Consequently, passing to Poincaré series we have

\[
P_{\Sigma_4} = \frac{1}{(1 - t^2)(1 - t^3)} + \frac{1}{(1 - t)^2} - \frac{1}{(1 - t)(1 - t^2)} = \frac{1 + t + t^2 + t^3}{(1 - t^2)(1 - t^3)}.
\]

Remark. Since \(H^*(D_8)\) is detected by abelian subgroups the same is true of \(H^*(\Sigma_4)\) and we have that \(H^*(\Sigma_4)\) is Cohen-Macaulay, i.e., freely and finitely generated over a polynomial subalgebra. In this case the subalgebra of elements which restrict to the Dickson algebras \(F_2[D_2, D_3]\) in both non-conjugate copies of \((\mathbb{Z}/2)^2 \subset \Sigma_4\).

Example (2.6). Example (2.5) generalizes to any extension of the form

\[1 \rightarrow V_{2n} \rightarrow G \rightarrow SL_2(F_{2n}) \rightarrow 0\]

by using the Tits building for \(SL_2(F_{2n})\). Once more \(X\) consists only of isolated points all in the same orbit and stabilizer the Borel group which, in this case, is given as the semi-direct product

\[B \cong (\mathbb{Z}/2)^n \rtimes (\mathbb{Z}/2^n - 1).\]

Hence (2.2) gives

\[H^*(G) \oplus H^*(V_{2n})^B \cong H^*(V_{2n})^{SL_2(F_{2n})} \oplus H^*(\text{Syl}_2(G))^{\mathbb{Z}/2^n - 1}.\]

From the geometry of the constructions above it is direct to see that this expression is more than merely a formal isomorphism. It corresponds to an exact sequence

\[0 \rightarrow H^*(G) \rightarrow H^*(V_{2n})^{SL_2(F_{2n})} \oplus H^*(\text{Syl}_2(G))^{\mathbb{Z}/2^n - 1} \rightarrow H^*(V_{2n})^B \rightarrow 0\]

which identifies \(H^*(G)\) as those elements \(\mu \in H^*(\text{Syl}_2(G))^{\mathbb{Z}/2^n - 1}\) which satisfy the condition

\[(i_{V_{2n}}^G \mu) \in H^*(V_{2n})^{SL_2(F_{2n})}.\]

Indeed, we could have stated Theorem (2.2) in terms of a split long exact sequence of the type above. This follows from the version of (2.1) which is proved in [W3]. In practice, however, we use (2.2) for Poincaré series calculations, usually preferring double coset decompositions to determine the ring structure of the resulting cohomology groups.

In the special case \(SL_2(F_4) \cong A_5\), the invariants \(H^*(V_4)^{A_5}\) are studied in [AM1] as they play an important role in the calculation of \(H^*(L_3(4))\) and the cohomology of the sporadic simple groups \(J_2, J_3, M_{22},\) and \(O'N\) as discussed in the introduction.
Example (2.7). There is a non-split extension of the form

\[ 1 \rightarrow (\mathbb{Z}/2)^3 \rightarrow E \rightarrow GL_3(\mathbb{F}_2) \rightarrow 1 \]

where the action of $GL_3(\mathbb{F}_2)$ on $(\mathbb{Z}/2)^3$ is the usual one. Indeed, Alperin [A] has given a complete discussion of extensions of the form $(\mathbb{Z}/2r)^3 \cdot GL_3(\mathbb{F}_2)$ where the action is the usual one when restricted to $(\mathbb{Z}/2)^3 \subset (\mathbb{Z}/2r)^3$, and has shown that there are exactly two, the first which is split and one other which is non-split. The group $E$ is the basic non-split example and is particularly interesting because $Syl_2(E) \cong Syl_2(M_{12})$, (compare (4.2)).

Using the Tits building for $GL_3(\mathbb{F}_2)$ with parabolics $P_1 \cong \Sigma_4$, $P_2 \cong \Sigma_4$ and $B = P_1 \cap P_2 \cong D_8$ we obtain

\[ H^*(E) \cong H^*(Syl_2(E)) \oplus H^*(V_3)^P_1 \oplus H^*(V_3)^P_2 \]

\[ \cong H^*(V_3)^{GL_3(\mathbb{F}_2)} \oplus H^*(\pi^{-1}(P_1)) \oplus H^*(\pi^{-1}(P_2)) \oplus H^*(V_3)^{D_8}. \]

Example (2.8). Consider the group $A_6 \subseteq Sp_4(\mathbb{F}_2) \cong \Sigma_6$. The resulting action of $A_6$ on $V_4$ allows us to define a semi-direct product

\[ G = V_4 \times A_6 \]

which is a 2-local subgroup of the third Mathieu group $M_{22}$. In this case (2.1) reduces modulo projectives to

\[ F_2 \oplus F_2[A_6/D_8] \cong F_2[A_6/Q_1] \oplus F_2[A_6/Q_2], \]

where $Q_1 \cong Q_2 \cong \Sigma_4$ are the two distinct copies of $\Sigma_4$ contained in $A_6$. We have

\[ H^*(G) \cong H^*(Syl_2(G)) \oplus H^*(V_4)^{Q_1} \oplus H^*(V_4)^{Q_2} \]

\[ \cong H^*(V_4)^{A_6} \oplus H^*(V_4 \times Q_1) \oplus H^*(V_4 \times Q_2) \oplus H^*(V_4)^{D_8}. \]

The invariants $H^*(V_4)^{A_6}$ will be determined in §3.

3. The invariant subrings for subgroups of $GL_4(\mathbb{F}_2)$

The classical isomorphism of $GL_4(2)$ with $A_8$ is given explicitly in [D, pp.290-292] by setting

\[ E_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \]

\[ E_4 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad E_5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad E_6 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \]
verifying the relations

\[ E_1^3 = E_1^{i+1} = (E_i E_{i+1})^3 = (E_i E_j)^2 = 1, \quad (i, j = 1, \ldots, 6, j > i + 1), \]

and setting up the correspondence

\[ E_1 \sim (123), \quad E_2 \sim (12)(34), \quad E_3 \sim (12)(45). \]

If we make a change of basis

\[ e_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ \end{pmatrix}, \]

then the first four matrices take the simpler form

\[
E_1 \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ \end{pmatrix}, \quad E_2 \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \end{pmatrix},
\]

\[
E_3 \sim \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \end{pmatrix}, \quad E_4 \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ \end{pmatrix}.
\]

Consequently, if we set \( \zeta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \) we obtain

\[ E_1 \sim \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{pmatrix}, \quad E_2 \sim \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad E_3 \sim \begin{pmatrix} J & I \\ 0 & J \end{pmatrix}, \quad E_4 \sim \begin{pmatrix} J & 0 \\ I & J \end{pmatrix}. \]

It follows that \( E_3 E_4 E_3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \) and the three matrices \( (E_1, E_2, E_3 E_4 E_3) \) are our usual generators for \( SL_2(\mathbb{F}_4) = A_5 \). Finally adding \( E_4 \) gives generators for the commutator subgroup of the symplectic group \( Sp_4(\mathbb{F}_2) \cong \Sigma_6, \quad Sp_4(\mathbb{F}_2)' = A_6. \)

We studied certain of these subgroups in [AM1]. In particular we studied the first \( \Sigma_4 \) and \( SL_2(\mathbb{F}_4) \) there. The essential step was to use cohomology with \( \mathbb{F}_4 \) as coefficient ring so that the elements of order three could be diagonalized. In particular the main process was as follows. Set \( A = \zeta_3 a + \zeta_3^2 b, \quad B = \zeta_3^2 a + \zeta_3 b, \quad C = \zeta_3 c + \zeta_3^2 d, \quad D = \zeta_3^2 c + \zeta_3 d \) where \( a, b, c, d \in Hom(\mathbb{F}_2^4, \mathbb{F}_2) \) are dual to \( e_1, e_2, e_3, e_4 \) respectively. Then the action of our generators above (or more accurately their duals) is given by

\[ E_1^*(A) = \zeta_3 A, \quad E_1^*(B) = \zeta_3^2 B, \quad E_1^*(C) = \zeta_3^2 C, \]
\[ E_5^*(A) = A, \quad E_5^*(B) = B, \quad E_5^*(C) = A + C, \quad E_5^*(D) = B + D, \]

and setting \( \tau = E_3 E_4 E_3 \), then

\[ \tau^*(A) = C, \quad \tau^*(B) = D. \]

Finally, in terms of this basis the action of the new generator is given by the formula

\[ E_4^*(A) = B, \quad E_4^*(B) = A. \]

Incidentally, if we set \( E_{12} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) then we see that with respect to the new basis,

\[ E_5 \sim \begin{pmatrix} E_{12} & 0 \\ I & E_{12} \end{pmatrix}, \quad E_6 \sim \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}. \]

Note that \( \tau = E_3 E_4 E_3 \sim (12)(46) \) under the correspondence with \( A_6 \) so the span \( (E_1, E_2, \tau, E_6) \cong \Sigma_5 \). This copy of \( \Sigma_5 \) is obtained by extending the action of \( SL_2(\mathbb{F}_4) \) via the Galois automorphism of \( \mathbb{F}_4 \). It also embeds in the full symplectic group \( Sp_4(\mathbb{F}_2) \).

**The \( \Sigma_4 \) subgroups of \( A_6 \)**

There are two non-conjugate copies of \( \Sigma_4 \subset A_6 \), the respective normalizers of the two non-conjugate copies of \( (\mathbb{Z}/2)^2 \) there.

The first, generated by \( (E_1, E_2, E_4) \) is characterized by the fact that the element \( E_1 \) of order 3 has no fixed non-zero vectors in \( (\mathbb{Z}/2)^4 \). The second is generated by

\[
S \quad = \quad E_1 E_3 E_4 \sim (123)(465) \\
T \quad = \quad E_2 E_3 E_4^2 E_2 E_1 E_3 E_2 \sim (15)(24) \\
E_3 \quad \sim \quad (12)(45)
\]

For this subgroup \( \text{Dim}((\mathbb{Z}/2)^4)^S) = 2 \), and defining new coordinates by

\[
A = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad D = \begin{pmatrix} 6 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\]

the original matrices become

\[
S \sim \begin{pmatrix} 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix}, \quad T \sim \begin{pmatrix} 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix}
\]
The invariant subring for the first $E_4$ is discussed in [AM1], see in particular (4.10), so we don’t discuss it further here. We now describe the invariants of the second $E_4 = (S, T, E_3)$. We set $w_i$ equal to the $i$th symmetric monomial in the variables $(x_1, \ldots, x_n)$. Note that, while the $w_i$, $1 \leq i \leq 4$ are not invariants of this $E_4$ action, they are invariants for the action of the subgroup $A_4$, since the change of basis above shows that $A_4 = (S, T)$ acts by the usual permutation of coordinates.

The following result is contained in [H].

**LEMMA (3.2).** Let $A_n$ act on $(x_1, \ldots, x_n)$ by permuting coordinates in the usual manner, then

$$F_2[x_1, \ldots, x_n]^{A_n} = F_2[w_1, \ldots, w_n](1, a_{n(n-1)/2})$$

where $a_{n(n-1)/2} = \sum_{\sigma \in A_n} \sigma(x_1 x_2^2 x_3^3 \ldots x_{n-1}^{n-1})$.

Now we are able to state

**THEOREM (3.3).** Let $\bar{w}_3 = w_3 + w_1 w_2$, $\gamma_4 = w_2(w_2 + w_1^2)$, $\gamma_8 = w_4(w_4 + w_3 w_1)$, $\lambda_5 = Sq^2(\bar{w}_3) = w_1 w_4 + (w_2 + w_1^2)\bar{w}_3$, and $b_6 = a_6 + w_2(\gamma_4 + w_1^4 + w_4) + w_1 w_2 w_3$. Finally, set $b_7 = w_1 b_6 + w_4 \bar{w}_3$. Then

$$F_2[x_1, x_2, x_3, x_4]^{E_4} = F_2[w_1, \bar{w}_3, \gamma_4, \gamma_8](1, \lambda_5, b_7, \lambda_5 b_7)$$

where the action of $E_4$ is determined by the matrices for $S, T, E_3$ above.

**Proof.** We begin by noting that $E_3$ normalizes $A_4$ so that it acts on

$$F_2[w_1, w_2, w_3, w_4](1, a_6),$$

and the desired invariants are the elements in this ring which are fixed under $E_3$. We have

**LEMMA (3.4).**

$$
\begin{align*}
E_3^* (w_1) & = w_1 \\
E_3^* (w_2) & = w_2 + w_1^2 \\
E_3^* (w_3) & = \bar{w}_3 \\
E_3^* (w_4) & = w_4 + w_1 \bar{w}_3 \\
E_3^* (a_6) & = a_6 + w_1^2(\gamma_4 + w_1^4 + w_4) + w_3 \bar{w}_3.
\end{align*}
$$
Proof (3.4). Since $E_3 = E_3^*$ we have $E_3^*(x_i) = x_{j(i)} + w_1$ for appropriate $j(i)$. Consequently $E_3^*(w_1) = 5w_1 = w_1$. Similarly $E_3^*(x_j) = x_a x_b + (x_a + x_b) w_1 + w_1^2$, and summing over all pairs $i,j$ we get all pairs $a,b$. Similarly $E_3^*(w_3) = w_3 + w_1^2$, and the above result follows for $w_4$.

The result for $a_6$ was done using a computer.

COROLLARY (3.5). We have that

$$(E_3^* + 1)a_6 + w_2(\gamma_4 + w_4^4 + w_4 + w_1 w_3) = \bar{w}_3^2.$$ 

We now return to the proof of (3.3). We can write $\mathbb{F}_2[x_1, \ldots, x_4]^A = \mathbb{F}_2[w_1, \ldots, w_4](1, b_6)$ and

$$\mathbb{F}_2[w_1, w_2, w_3, w_4] = \mathbb{F}_2[w_1, w_2, w_3, w_4] = \mathbb{F}_2[w_1, w_3, \gamma_4, \gamma_6](1, w_2, w_4, w_2 w_4).$$

We set $S = \mathbb{F}_2[w_1, w_3, \gamma_4, \gamma_6]$ and

$$S_1 = S(1, w_2, w_4),$$

which is closed under the action of $E_3$. There is an exact sequence of $E_3$ modules

$$S \rightarrow S_1 \rightarrow S\{w_2\} \oplus S\{w_4\},$$

which leads to the long exact sequence

$$S \rightarrow S_{E_3} \rightarrow S\{w_2\} \oplus S\{w_4\} \rightarrow H^1(\mathbb{Z}/2;S) = S \rightarrow H^1(\mathbb{Z}/2;S_1) \rightarrow \cdots.$$ 

Clearly $\delta(\{w_2\}) = w_1^2$ while $\delta(\{w_4\}) = w_1 \bar{w}_3$ so $\text{Ker}(\delta) = S(\lambda_5)$. In the $H^1$ part of this exact sequence we note for future use that the exact sequence becomes

$$0 \rightarrow S/(w_1^2, w_1 \bar{w}_3) \rightarrow H^1(\mathbb{Z}/2;S) \rightarrow S\lambda_5 \rightarrow 0.$$ 

Next, set $S_2 = S(1, w_2, w_4, w_2 w_4) = \mathbb{F}_2[w_1, w_2, w_3, w_4]$. We have an exact sequence

$$S_1 \rightarrow S_2 \rightarrow S\{w_2 w_4\},$$

which gives us the long exact sequence

$$S_{E_3} \rightarrow S_2 \rightarrow S\{w_2 w_4\} \rightarrow H^1(\mathbb{Z}/2;S_1) \rightarrow \cdots.$$ 

Moreover, an easy check gives that $\delta(\{w_2 w_4\}) = w_1 \lambda_5$, so $\delta$ is an injection and $S_{E_3}$ is the entire invariant subring. This also gives us $H^1(\mathbb{Z}/2;S_2)$.

Finally, we add $b_6$. We have the exact sequence

$$0 \rightarrow S_2 \rightarrow S_2(1, b_6) \rightarrow S_2\{b_6\} \rightarrow 0,$$
and passing to cohomology we get the long exact sequence

\[ S_2^E \rightarrow S_2(1, b_6)^E \rightarrow S_2^E\{b_6\} \xrightarrow{\delta} H^1(\mathbb{Z}/2; S_2) \rightarrow \ldots. \]

Here the corollary above shows that \( \delta(\{b_6\}) = \tilde{w}_3^2 \), and our calculation of 
\[ H^1(S_2; \mathbb{Z}/2) \] shows that \( \text{Ker}(\delta) = S_2^E\{b_6\} \). But the class \( w_1b_6 + w_4\tilde{w}_3 \) is an actual \( E_3 \)-invariant, so it becomes \( b_7 \) and we have proved the result.

**The two \( A_5 \)'s**

There are two copies of \( A_5 \) in \( A_6 \). The first, given as \( (E_1, E_2, E_3 E_4 E_3) \), is discussed in [AM1] and its invariant subring is determined there. The other is studied as follows. First, from the Atlas, [C], it is direct to check that there are only two non-conjugate copies of \( A_5 \subset A_8 \), and consequently the second group may be realized from the usual action of \( A_5 \) on \( \mathbb{F}_2[x_1, \ldots, x_5] \) by projection onto the \( \Sigma_5 \) invariant subring generated by the four elements

\[ r_1 = x_1 + x_2, \quad r_2 = x_1 + x_3, \quad r_3 = x_1 + x_4, \quad \text{and} \quad r_4 = x_1 + x_5. \]

The remaining element in this basis is \( x_1 + x_2 + x_3 + x_4 + x_5 = w_1 \), and rewriting

\[ \mathbb{F}_2[x_1, \ldots, x_5]_{A_5} = \mathbb{F}_2[w_1] \otimes \mathbb{F}_2[r_1, \ldots, r_4]_{A_5}. \]

Consequently, we can apply Hewett's result so

\[ \mathbb{F}_2[r_1, \ldots, r_4]_{A_5} \cong \mathbb{F}_2[v_2, v_3, v_4, v_5](1, b_{10}) \]

and arguing similarly,

\[ \mathbb{F}_2[r_1, \ldots, r_4]_{A_5} \cong \mathbb{F}_2[v_2, v_3, v_4, v_5] \]

where we can easily determine the explicit forms of the \( v_i \).

**The \( A_6 \) invariant subring**

The Dickson element \( D_8 \) is given as follows in terms of the above generators for \( \mathbb{F}_2[x_1, \ldots, x_4]_{A_4} \).

\[ D_8 = \gamma_8 + \gamma_4 \alpha + w_1^8 + w_1^2w_3^2. \]

Also, the Dickson element \( D_{12} = Sq^4(D_8) \) has the following representation

\[ r_{12} D_{12} = (D_8 + (w_1\tilde{w}_3 + \gamma_4 + w_1^4\alpha)(\gamma_4 + w_1\tilde{w}_3 + w_1^4) + (w_1\lambda_5 + w_1^2\gamma_4 + w_3^2)^2. \]

Finally, there is the relation

\[ r_{10} = \lambda_5^2 = w_1^2\tilde{w}_3\lambda_5 + w_1^2D_8 + \tilde{w}_3^2(w_1^4 + \gamma_4) + \gamma_4^2w_1^2 + w_1^2. \]

**Lemma (3.6).** The class \( \tilde{w}_3 \) satisfies \( E_1(\tilde{w}_3) = \tilde{w}_3 \) so \( \tilde{w}_3 \in \mathbb{F}_2[x_1, \ldots, x_4]_{A_5} \).

Also, \( \tilde{w}_3, \quad Sq^2(\tilde{w}_3) = \lambda_5, \quad D_8, \quad \text{and} \quad D_{12} \) are transcendentally independent so \( \mathbb{F}_2[x_1, \ldots, x_4]_{A_5} \) contains the polynomial algebra

\[ \mathbb{F}_2[\tilde{w}_3, \lambda_5, D_8, D_{12}]. \]
Proof. \( \bar{w}_3 = S(x_1 x_2) \) gives a representation of \( \bar{w}_3 \) as a symmetric sum. Now, \( E_1^* \) is determined by the formula

\[
\begin{align*}
   x_1 &\mapsto (x_2 + x_4) \\
   x_2 &\mapsto (x_3 + x_4) \\
   x_3 &\mapsto (x_1 + x_4) \\
   x_4 &\mapsto (x_1 + x_2 + x_3)
\end{align*}
\]

We can write \( S(x_1^2 x_2) = x_1^2(x_2 + x_3 + x_4) + x_1(x_2 + x_3 + x_4)^2 + x_2^2(x_3 + x_4) + x_2(x_3 + x_4)^2 + x_3^2 x_4 + x_3 x_4^2 \). Consequently

\[
E_1^*(\bar{w}_3) = (x_2 + x_4)^2 x_2 + (x_2 + x_4) x_2^2 \\
+ (x_3 + x_4)^2 (x_2 + x_3 + x_4) + (x_3 + x_4) (x_2^2 + x_3^2 + x_4^2) \\
+ (x_1 + x_4)^2 (x_1 + x_2 + x_3) + (x_1 + x_4) (x_1^2 + x_2^2 + x_3^2),
\]

and one sees directly that this is again \( S(x_1^2 x_2) \). The transcendental independence of the generators above is evident by inspection.

**COROLLARY (3.7).** The 9 dimensional class \( \lambda_9 = Sq^4(\lambda_5) \) is also invariant under \( A_6 \).

The following result is our main technique in determining \( F_2[x_1, \ldots, x_4]^{A_6} \) from \( F_2[x_1, \ldots, x_4]^{E_4} \).

**LEMMA (3.8).** There exists an explicit projection operator

\[
e : F_2[x_1, \ldots, x_4]^{E_4} \rightarrow F_2[x_1, \ldots, x_4]^{A_6},
\]

i.e., \( e^2 = id \) restricted to the \( A_6 \) invariant subring, and \( \text{im}(e) \subset F_2[x_1, \ldots, x_4]^{A_6} \).

Proof. We write \( A_6 = \bigsqcup_{i=1}^{15} v_i \Sigma_4 \) for an explicit coset decomposition of \( A_6 \). Then we set

\[
e = \sum_{1}^{15} v_i.
\]

Clearly, if \( \alpha \in F_2[x_1, \ldots, x_4]^{A_6} \) we have \( e(\alpha) = 15 \alpha = \alpha \), while for \( \alpha \in F_2[x_1, \ldots, x_4]^{E_4} \) and \( g \in A_6 \) we have \( g v_i = v_{g(i)} s_i \) with \( s_i \in \Sigma_4 \) so \( g v_i(\alpha) = v_{g(i)}(\alpha) \) and \( g(e(\alpha)) = e(\alpha) \).

**Remark.** The situation in (3.8) occurs for any subgroup \( H \subset A_6 \), as long as \( H \) contains a Sylow 2-subgroup of \( A_6 \).

Using (3.8) we obtain

**THEOREM (3.9).** The ring of invariants \( F_2[x_1, \ldots, x_4]^{A_6} \) is

\[
F_2[\bar{w}_3, \lambda_5, D_8, D_{12}]\langle 1, \lambda_9, b_{15}, \lambda_9 b_{15}\rangle
\]
where \( b_{15} = e(\gamma_4^2 b_7) \).

**Proof.** The major step in the proof is to reduce the determination of the \( A_6 \) invariants to a finite calculation. This is a direct consequence of

**Lemma (3.10).** \( \FF_2[x_1, \ldots, x_4]^{S_4} \) is freely generated over the \( A_6 \)-invariant polynomial subring,
\[
E = \FF_2[\tilde{w}_3, \lambda_5, D_8, D_{12}]
\]
on the 60 generators
\[
(1, b_7) \quad \left\{ \begin{align*}
1, w_1, w_1^2, w_1^3, \ldots, w_1^9 \\
\gamma_4, w_1 \gamma_4, w_1^2 \gamma_4, \ldots, w_1^9 \gamma_4 \\
\gamma_4^2, w_1 \gamma_4^2, \ldots, w_1^5 \gamma_4 \\
w_1^2 \gamma_4, w_1^3 \gamma_4, w_1^4 \gamma_4, w_1^5 \gamma_4.
\end{align*} \right.
\]

**Proof of (3.10).** We have that the Poincaré series for the free \( B \)-module on 60 generators having the dimensions of the generators above is
\[
\frac{(1 + x^7)(1 + x^4 + x^8)(1 - x^{10})}{(1 - x)(1 - x^3)(1 - x^5)(1 - x^8)(1 - x^{12})} = \frac{(1 + x^5)(1 + x^7)}{(1 - x)(1 - x^3)(1 - x^4)(1 - x^5)}
\]
which is the Poincaré series for \( \FF_2[x_1, \ldots, x_4]^{S_4} \). Hence, if we can show that the \( B \)-submodule of \( \FF_2[x_1, \ldots, x_4]^{S_4} \) generated by the 60 elements above is the entire invariant ring the lemma will follow.

Let \( A = \FF_2[w_1, \tilde{w}_3, \gamma_4, D_8, D_{12}] \) \( (\lambda_5) \subset \FF_2[x_1, \ldots, x_4]^{S_4} \). We now wish to determine \( \text{Tor}_0^g(A, \FF_2) \) which determines a generating set for \( A \) over \( B \).

Let \( \mathcal{R} = \FF_2[w_1, \tilde{w}_3, \gamma_4, \lambda_5, D_8, D_{12}] \) be the polynomial algebra on (formal) generators. There is an obvious surjective map \( \mathcal{R} \rightarrow A \) and the kernel is the ideal generated by the two elements
\[
R_{12} = D_{12} + (D_8 + (w_1 \tilde{w}_3 + \gamma_4 + w_1^4)^2)(\gamma_4 + 3 \tilde{w}_3 w_1 + w_1^4) + (w_1 \lambda_5 + w_1^2 \gamma_4 + \tilde{w}_3^2),
\]
and
\[
R_{10} = \lambda_5^2 + w_1^2 \tilde{w}_3 \lambda_5 + w_1^{10} + (w_1^4 + \gamma_4) \tilde{w}_3^2 + (D_8 + \gamma_4^2) w_1^2.
\]
Consequently we obtain a resolution of \( A \) over \( \mathcal{R} \),
\[
\begin{array}{c}
0 \rightarrow \mathcal{R}s_{22} \xrightarrow{\partial} \mathcal{R}(r_{10}, r_{12}) \xrightarrow{\partial} \mathcal{R} \rightarrow A \rightarrow 0
\end{array}
\]
where \( \partial(r_{10}) = R_{10} \), \( \partial(r_{12}) = R_{12} \), and \( \partial(s_{22}) = R_{12}r_{10} + R_{10}r_{12} \).

Since \( \mathcal{R} \) is free over \( B \) the resolution (3.11) is also a resolution of \( A \) over \( B \). Moreover
\[
\mathcal{R}/B = \FF_2[w_1, \gamma_4]
\]
so the complex for computing \( \text{Tor}_0^g(A, \FF_2) \) becomes
\[
\begin{array}{c}
\FF_2[w_1, \gamma_4]s_{22} \xrightarrow{\partial} \FF_2[w_1, \gamma_4](r_{10}, r_{12}) \xrightarrow{\partial} \FF_2[w_1, \gamma_4] \rightarrow \text{Tor}_0^g(A, \FF_2) \rightarrow 0.
\end{array}
\]
Here $\partial(r_{10}) = w_{10}^0 + \gamma_2^2 w_1^2$ and $\partial(r_{12}) = (\gamma_4 + w_1^4)^3 + w_1^4 \gamma_4^2$. Now a direct calculation gives that $\text{Tor}_6^S(A, \mathbb{F}_2)$ has 30 generators,

$$\text{Tor}_6^S(A, \mathbb{F}_2) = \left\{ 1, w_1, w_1^2, w_1^3, \ldots, w_1^9, \gamma_4, w_1^{174}, w_1^{174} w_1^2, \ldots, w_1^{174} \gamma_4, \gamma_4^2, w_1^{174} \gamma_4^2, \ldots, w_1^{174} \gamma_4^2, w_1^{2,3}, w_1^{2,3} w_1^{174}, w_1^{2,3} w_1^{174} w_1^{174} \right\}$$

and it follows that $F_2[x_1, \ldots, x_4]^S = A(1, \delta_8)$ is free over $B$ on the sixty generators asserted in (3.10).

The proof of (3.9) is now a direct computer computation. The projection operator $e$ is evaluated, in turn, on each of the sixty generators above. Note that $B$ and these images certainly generate $F_2[x_1, \ldots, x_4]^S$, so the final determination of the ring became simply a matter of identifying the independent generators among these images. Again this was done using the computer.

**The $A_7$ invariant subring**

The argument for $A_7$ is similar to that for $A_6$. We start with the Dickson algebra $D = F_2[D_8, D_{12}, D_{14}, D_{15}] = F_2[x_1, \ldots, x_4]^{GL_2(4)}$ and show to begin that $F_2[x_1, \ldots, x_4]^S$ is freely and finitely generated over it on 56 generators. Then we use a projector similar to the $e$ of the previous section,

$f: F_2[x_1, x_2, x_3, x_4]^A_6 \longrightarrow F_2[x_1, x_2, x_3, x_4]^A_7$

on each of the 56 generators above to obtain a generating set for $F_2[x_1, \ldots, x_4]^A_7$.

**Lemma (3.12).** There exists an explicit projection operator

$f: F_2[x_1, \ldots, x_4]^A_6 \longrightarrow F_2[x_1, \ldots, x_4]^A_7$,

i.e., $f^2 = \text{id}$ when restricted to the $A_7$ invariant subring and $\text{im}(f) \subset F_2[x_1, \ldots, x_4]^A_7$.

The proof is exactly like that of (3.8). Once more $|A_7 : A_6| = 7$ is odd and so $\sum_i w_i = f$ where the $w_i$ are a set of coset generators for $A_7$ over $A_6$.

**Theorem (3.13).** $F_2[x_1, \ldots, x_4]^A_7$ is freely and finitely generated over the Dickson algebra $D$ on eight generators, \{1\}, $f(\bar{w}_3^3 \lambda_9)$ in dimension 18, $f(\bar{w}_3^5 \lambda_6)$ in dimension 20, $f(\bar{w}_3^5 \lambda_6)$ in dimension 21, $f(\bar{w}_3^5 \lambda_6)$ in dimension 24, $f(\bar{w}_3^5 \lambda_6)$ in dimension 25, $f(\bar{w}_3^6 \lambda_9)$ in dimension 27, and $f(\bar{w}_3^7 \lambda_9 \lambda_15)$ in dimension 45. In particular it has Poincaré series

$$\frac{1 + x^{18} + x^{26} + x^{21} + x^{24} + x^{25} + x^{27} + x^{45}}{(1 - x^8)(1 - x^{12})(1 - x^{14})(1 - x^{15})}.$$
The proof follows that of the previous theorem on the $A_6$ invariant subring quite closely. We begin with

**Lemma (3.14).** The subring of $\mathbb{F}_2[x_1, \ldots, x_4]^{A_6}$ given as $B = \mathbb{F}_2[\bar{w}_3, \lambda_5, D_8, D_{12}] (1, \lambda_9)$ contains the Dickson algebra $\mathcal{D}$ and is freely and finitely generated over $\mathcal{D}$ on the 28 generators

$$\begin{align*}
1, \bar{w}_3, \bar{w}_5^3, \bar{w}_3^3, \bar{w}_3^5, \bar{w}_3^7, \\
\lambda_9, \lambda_9 \bar{w}_3, \lambda_9 \bar{w}_3^3, \lambda_9 \bar{w}_3^5, \lambda_9 \bar{w}_3, \lambda_9 \bar{w}_3^7, \\
\lambda_5, \lambda_5 \bar{w}_3, \lambda_5 \bar{w}_3^3, \lambda_5 \bar{w}_3^5, \lambda_5 \bar{w}_3, \lambda_5 \bar{w}_3^7, \\
\lambda_5, \lambda_5^2 \bar{w}_3, \lambda_5^2 \bar{w}_3^3, \lambda_5^2 \bar{w}_3^5, \lambda_5^2 \bar{w}_3, \lambda_5^2 \bar{w}_3^7.
\end{align*}$$

Consequently, $\mathbb{F}_2[x_1, \ldots, x_4]^{A_6}$ is free and finitely generated over $\mathcal{D}$ on the 56 generators consisting of the 28 generators above and their products with $\lambda_{15}$.

**Proof of (3.14).** One checks directly that we have

$$\begin{align*}
D_{14} &= \lambda_5 \lambda_9 + \bar{w}_3^2 D_8 + \bar{w}_3^3 \lambda_5, \\
D_{15} &= \lambda_5^3 + \bar{w}_3^2 \lambda_9 + \bar{w}_3^5.
\end{align*}$$

This shows that $\mathcal{D} \subset B$.

Let $\mathcal{C} = \mathbb{F}_2[\bar{w}_3, \lambda_5, \lambda_9, D_8, D_{12}, D_{14}, D_{15}]$ be the polynomial algebra on 7 generators in the stated dimensions. There is a surjection $\mathcal{C} \rightarrow B$ taking the generators to their images with the same names. To determine a resolution of $B$ over $\mathcal{C}$ note the three relations

$$\begin{align*}
R_{18}: &\quad \lambda_5^2 + D_8 \lambda_5^2 + \bar{w}_3^6 + \lambda_9^3 \bar{w}_3^3 + \lambda_9 \bar{w}_3^3 + D_{12} \bar{w}_3^2 \\
R_{15}: &\quad D_{15} + \bar{w}_3^3 \lambda_9 + \lambda_5^3 + \bar{w}_3^5 \\
R_{14}: &\quad D_{14} + \lambda_5 \lambda_9 + \bar{w}_3^2 D_8 + \bar{w}_3^3 \lambda_{15}.
\end{align*}$$

We clearly obtain a resolution of $B$ over $\mathcal{C}$ as

$$0 \rightarrow \mathcal{C}(s_{47}) \rightarrow \mathcal{C}(s_{29}, s_{32}, s_{33}) \rightarrow \mathcal{C}(s_{14}, s_{15}, s_{18}) \rightarrow \mathcal{C} \rightarrow B \rightarrow 0$$

where $\partial(s_{14}) = R_{14}$, $\partial(s_{15}) = R_{15}$, $\partial(s_{16}) = R_{18}$, and so on. Consequently, since $\mathcal{C}$ is free over the Dickson algebra $\mathcal{D}$ we obtain a resolution of $B$ over $\mathcal{D}$ as follows. Set

$$\mathcal{E} = \mathbb{F}_2[\bar{w}_3, \lambda_5, \lambda_9] \subset \mathcal{C},$$

then a chain complex for determining $\text{Tor}_*^\mathcal{D}(B, \mathbb{F}_2)$ is given as

$$\mathcal{E}(s_{47}) \rightarrow \mathcal{E}(s_{29}, s_{32}, s_{33}) \rightarrow \mathcal{E}(s_{14}, s_{15}, s_{18}) \rightarrow \mathcal{E} \rightarrow \text{Tor}_0^\mathcal{D}(B, \mathbb{F}_2) \rightarrow 0$$

where $\partial(s_{14}) = R_{14}$, $\partial(s_{15}) = R_{15}$, $\partial(s_{16}) = R_{18}$, and so on. Consequently, since $\mathcal{C}$ is free over the Dickson algebra $\mathcal{D}$ we obtain a resolution of $B$ over $\mathcal{D}$ as follows. Set

$$\mathcal{E} = \mathbb{F}_2[\bar{w}_3, \lambda_5, \lambda_9] \subset \mathcal{C},$$

then a chain complex for determining $\text{Tor}_*^\mathcal{D}(B, \mathbb{F}_2)$ is given as

$$\mathcal{E}(s_{47}) \rightarrow \mathcal{E}(s_{29}, s_{32}, s_{33}) \rightarrow \mathcal{E}(s_{14}, s_{15}, s_{18}) \rightarrow \mathcal{E} \rightarrow \text{Tor}_0^\mathcal{D}(B, \mathbb{F}_2) \rightarrow 0$$
where \( \partial(s_{14}) = \lambda_5 \lambda_9 + \bar{w}_3^2 \lambda_5 \), \( \partial(s_{15}) = \bar{w}_3^2 \lambda_9 + \lambda_5^6 + \bar{w}_3^5 \), and \( \partial(s_{18}) = \lambda_5^6 + \lambda_3^2 \bar{w}_3 + \bar{w}_3^3 \lambda_9 + \bar{w}_3^6 \), and we have

\[
\text{Tor}_0^D(\mathcal{B}, \mathbb{F}_2) = \mathbb{F}_2[\bar{w}_3, \lambda_5, \lambda_9]/(\lambda_5 \lambda_9 + \bar{w}_3^2 \lambda_5, \bar{w}_3^2 \lambda_9 + \lambda_3^2 \bar{w}_3 + \bar{w}_3^3 \lambda_9 + \bar{w}_3^6).
\]

Within the ideal which is being factored out note the following. First the relation for \( \lambda_5^6 \) has the form \( \lambda_5^6 + \bar{w}_3 r_{15} \) where \( r_{15} \) is the second relation above. Thus \( \lambda_5^6 \in \) the relation set. Similarly, \( \lambda_9 r_{15} = \lambda_5^4 + \bar{w}_3^2 r_{14} \) so \( \lambda_5^6 \) is in the relation set. Also, \( \lambda_9 r_{14} = \bar{w}_3^2 \lambda_9 \lambda_9 + \lambda_5 ^3 \lambda_5 \) so \( \bar{w}_3^2 \lambda_9 \lambda_9 \) is in the relation set, and expanding out \( \lambda_5^3 r_{14} \) gives that \( \bar{w}_3^2 \lambda_9 = 0 \). Also, note that \( r_{14} \) implies that \( \bar{w}_3 \lambda_5 ^3 \lambda_5 + r_{15} \) implies that \( \bar{w}_3 + \bar{w}_3^2 \lambda_9 + \lambda_5 ^3 \). Consequently

\[
\bar{w}_3 r_{15} = \bar{w}_3^5 \lambda_9 + \lambda_3^2 \lambda_9 + \bar{w}_3^4 = (\bar{w}_3^2 \lambda_9 + \lambda_3^2) \lambda_9 + \lambda_5^3 \lambda_9 + \bar{w}_3^3 = \bar{w}_3^3
\]

and \( \bar{w}_3^3 \) is in the relation set. Thus, \( \text{Tor}_0^D(\mathcal{B}, \mathbb{F}_2) \) can be no larger than the set asserted in (3.14).

We now show that it also cannot be smaller. Note that the Poincaré series for the free module over \( \mathcal{D} \) on generators in the stated dimensions above is

\[
\frac{1-x^{24}}{1-x^5}(1+x^9) + \frac{1-x^{18}}{1-x^7}(x^5 + x^{10})
\]

but this is

\[
\frac{(1+x^9)(1-x^{14})(1+x^5+x^{10})}{(1-x^5)(1-x^9)(1-x^{12})(1-x^{14})(1-x^{15})} = \frac{1+x^9}{(1-x^3)(1-x^6)(1-x^8)(1-x^{12})}
\]

and (3.14) follows.

The next step in the proof is contained in

**Lemma (3.15).** \( \mathbb{F}_2[x_1, \ldots, x_4]^{\mathcal{A}_7} \) is freely and finitely generated over \( \mathcal{D} \) on eight generators.

**Proof of (3.15).** An easy induction using (3.14) and the projector \( f \) shows that the ring of invariants \( \mathbb{F}_2[x_1, \ldots, x_4]^{\mathcal{A}_7} \) is freely and finitely generated over \( \mathcal{D} \). It remains to show that the number of generators is eight. To see this we pass to quotient fields. We have that the degree of \( \mathbb{F}_2[x_1, \ldots, x_4]^{\mathcal{A}_7} \) over \( \mathbb{F}_2(x_1, \ldots, x_4)^{GL_4(2)} \) is eight by the fundamental theorem of Galois theory. On the other hand \( \mathbb{F}_2(x_1, \ldots, x_4)^{GL_4(2)} = \mathcal{D} \) and thus \( \mathcal{QD} = \mathbb{F}_2(x_1, \ldots, x_4)^{GL_4(2)} \) so

\[
\mathcal{Q}([\mathbb{F}_2[x_1, \ldots, x_4]^{\mathcal{A}_7}]) = \mathcal{QD}(w_1, \ldots, w_8)
\]

where \( w_1, \ldots, w_8 \) are the eight generators of \( \mathbb{F}_2[x_1, \ldots, x_4]^{\mathcal{A}_7} \) over \( \mathcal{D} \).
For the final step the operator $f$ was evaluated on the 28 generators in the first lemma above using a computer, and image classes in dimensions 18, 20, 21, 24, 25, and 27 were found. The resulting table is given as follows where $S_{x^1}x^2x^3x_4$ denotes the symmetric sum

The class $f(w^3_{79})$:
$$\begin{align*}
S_{x^1}x^2x^3x_4 & + S_{x^1}x^2x^3x_4 + S_{x^1}x^2x^3x_4 \\
+ S_{x^1}x^2x^3x_4 & + S_{x^1}x^2x^3x_4 + S_{x^1}x^2x^3x_4 \\
+ S_{x^1}x^2x^3x_4 & + S_{x^1}x^2x^3x_4 + S_{x^1}x^2x^3x_4 \\
+ S_{x^1}x^2x^3x_4 & + S_{x^1}x^2x^3x_4 + S_{x^1}x^2x^3x_4
\end{align*}$$

The class $f(w^5_{79})$:
$$\begin{align*}
S_{x^1}x^2x^3x_4 & + S_{x^1}x^2x^3x_4 + S_{x^1}x^2x^3x_4 \\
+ S_{x^1}x^2x^3x_4 & + S_{x^1}x^2x^3x_4 + S_{x^1}x^2x^3x_4 \\
+ S_{x^1}x^2x^3x_4 & + S_{x^1}x^2x^3x_4 + S_{x^1}x^2x^3x_4 \\
+ S_{x^1}x^2x^3x_4 & + S_{x^1}x^2x^3x_4 + S_{x^1}x^2x^3x_4
\end{align*}$$

The class $f(w^2)$:
$$\begin{align*}
S_{x^1}x^2x^3x_4 & + S_{x^1}x^2x^3x_4 + S_{x^1}x^2x^3x_4 \\
+ S_{x^1}x^2x^3x_4 & + S_{x^1}x^2x^3x_4 + S_{x^1}x^2x^3x_4 \\
+ S_{x^1}x^2x^3x_4 & + S_{x^1}x^2x^3x_4 + S_{x^1}x^2x^3x_4 \\
+ S_{x^1}x^2x^3x_4 & + S_{x^1}x^2x^3x_4 + S_{x^1}x^2x^3x_4
\end{align*}$$

The class $f(w^5_{579})$:
$$\begin{align*}
S_{x^1}x^2x^3x_4 & + S_{x^1}x^2x^3x_4 + S_{x^1}x^2x^3x_4 \\
+ S_{x^1}x^2x^3x_4 & + S_{x^1}x^2x^3x_4 + S_{x^1}x^2x^3x_4 \\
+ S_{x^1}x^2x^3x_4 & + S_{x^1}x^2x^3x_4 + S_{x^1}x^2x^3x_4 \\
+ S_{x^1}x^2x^3x_4 & + S_{x^1}x^2x^3x_4 + S_{x^1}x^2x^3x_4
\end{align*}$$

The class $f(w^5_{575})$:
$$\begin{align*}
S_{x^1}x^2x^3x_4 & + S_{x^1}x^2x^3x_4 + S_{x^1}x^2x^3x_4 \\
+ S_{x^1}x^2x^3x_4 & + S_{x^1}x^2x^3x_4 + S_{x^1}x^2x^3x_4 \\
+ S_{x^1}x^2x^3x_4 & + S_{x^1}x^2x^3x_4 + S_{x^1}x^2x^3x_4 \\
+ S_{x^1}x^2x^3x_4 & + S_{x^1}x^2x^3x_4 + S_{x^1}x^2x^3x_4
\end{align*}$$
The class \( f(w_{79}) \): 
\[
\begin{align*}
&= Sx_{1}^{13} x_{3}^{4} x_{4}^{2} + Sx_{1}^{10} x_{2}^{3} x_{4}^{2} + Sx_{1}^{14} x_{2}^{3} x_{4}^{1} \\
&+ Sx_{1}^{15} x_{2}^{3} x_{3}^{2} + Sx_{1}^{9} x_{2}^{4} x_{3}^{2} + Sx_{1}^{12} x_{2}^{3} x_{4}^{3} \\
&+ Sx_{1}^{16} x_{3}^{3} x_{4}^{2} + Sx_{1}^{11} x_{2}^{4} x_{3}^{2} \\
&\end{align*}
\]

These classes are easily checked and seen to be independent over \( D \). Thus these seven classes, together with \{1\} and one other class freely generate the \( A_7 \) invariant subring over \( D \). To find the last class note that the numerator in the Poincaré series for a Cohen-Macaulay ring satisfies a symmetry condition of the form, the coefficient of \( x^n \) is equal to the coefficient of \( x^{l-n} \) for some fixed \( l \). Here, the only possibility for \( l \) is 45, and the result follows.

### 4. The cohomology of some sporadic simple groups

In this section we will provide explicit applications of our invariant calculations to determine the mod(2) cohomology of certain important simple groups. We begin with a "small" group.

**Example (4.1).** Set \( G = M_{11} \), the first Mathieu group having order \( 11 \times 10 \times 9 \times 8 = 7920 = 2^4 \times 3^2 \times 5 \times 11 \). \( M_{11} \) has 2-rank two with one conjugacy class of groups \((\mathbb{Z}/2)^2\) and one conjugacy class of involutions. From the Atlas, \( [C] \), \( N(2A) = 2 \Sigma_4 = GL_2(\mathbb{F}_3) \), and we can also check that \( N(\langle \mathbb{Z}/2 \rangle^2) = \Sigma_4 \). Thus the quotient \( A_2(M_{11})/M_{11} \) has the form

\[
\Sigma_4 \quad \text{GL}_2(\mathbb{F}_3) \\
\downarrow \quad D_5
\]

Apply (2.1) to obtain the formula

\[
H^*(M_{11}) \oplus H^*(D_5) = H^*(GL_2(3)) \oplus H^*(\Sigma_4)
\]

and substitute for \( H^*(\Sigma_4) \) using the formula in (2.5) to get

\[
H^*(M_{11}) \oplus H^*(V_2)^{\mathbb{Z}/2} \cong H^*(GL_2(3)) \oplus H^*(V_2)^{GL_d(\mathbb{F}_2)}.
\]
From Quillen's results, [Q2], the Poincaré series for $H^*(GL_2(\mathbb{F}_3))$ is

$$\frac{(1 + t)(1 + t^3)}{(1 - t^2)(1 - t^4)} = \frac{1 + t + t^2 + t^5 + t^4 + t^5}{(1 - t^3)(1 - t^4)}$$

and so the Poincaré series for $M_{11}$ is (as first computed in [Wo1], compare [BC])

$$\frac{(1 + t + t^2 + t^3 + t^4 + t^5) + (1 + t^2) - (1 + t^2)(1 + t + t^3)}{(1 - t^3)(1 - t^4)} = \frac{1 + t^5}{(1 - t^3)(1 - t^4)}.$$

The group $GL_2(\mathbb{F}_3)$ contains a Sylow 2-subgroup of $M_{11}$ and has its mod(2) cohomology detected on its elementary 2-subgroups. Consequently the same is true for $M_{11}$. It follows that

$$H^*(M_{11}; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2]^{GL_2(\mathbb{F}_2)} = \mathbb{F}_2[D_2, D_3],$$

and since there is only one element in each of the dimensions 3, 4, and 5 in this ring we have an independent proof of the result that $H^*(M_{11}) \cong \mathbb{F}_2[D_2, D_3]$. In particular note that $H^*(M_{11})$ is Cohen-Macaulay.

**Example (4.2).** We now consider the second Mathieu group $M_{12}$ of order 95040 = $2^6 \cdot 3^3 \cdot 5 \cdot 11$. Its poset space is much more complex since it has two distinct conjugacy classes of involutions as well as three distinct conjugacy classes of $(\mathbb{Z}/2)^3$'s and four conjugacy classes of $(\mathbb{Z}/2)^2$. For the details of its structure see [AMM2]. In particular, from [AMM2] we have that the Poincaré series for $H^*(M_{12}; \mathbb{F}_2)$ is

$$P_{M_{12}}(t) = \frac{1 + t^2 + 3t^3 + t^4 + 3t^5 + 4t^6 + 2t^7 + 4t^8 + 3t^9 + t^{10} + 3t^{11} + t^{12} + t^{14}}{(1 - t^4)(1 - t^6)(1 - t^7)}$$

and, in fact, $H^*(M_{12}; \mathbb{Z}/2)$ is Cohen-Macaulay over the polynomial subring $\mathbb{F}_2[D_4, D_6, D_7]$. Since $H^*(BG_2; \mathbb{F}_2) \cong \mathbb{F}_2[D_4, D_6, D_7]$ as well, this gives the indication of some connections between $M_{12}$ and the exceptional Lie group $G_2$. Of course $M_{12} \not\subseteq G_2$ so the connection is not nearly as simple as group inclusion. However, in $M_{12}$ there are two maximal subgroups of order 192. The first is the holomorph of the quaternion group $Q_8$ which we write $W$, and the second is a split extension

$$(\mathbb{Z}/4)^2 \times (\mathbb{Z}/2 \times \Sigma_3)$$

where the $\mathbb{Z}/2$ acts to invert elements in $(\mathbb{Z}/4)^2$. But $W'$ is also seen to be the extension of the elements of order 4 in the usual torus of $G_2(q)$ for $q \equiv 3, 5 \mod (8)$ by the Weyl group of $G_2$. These groups both contain $Syl_2(M_{12})$ so there is a configuration.
contained in $M_{12}$. In [AMM2] it is shown that this configuration completely determines $H^*(M_{12}; \mathbb{F}_2)$ as the intersection in $H^*(\text{Syl}_2(M_{12}))$ of the restriction images of $H^*(W)$ and $H^*(W')$. On the other hand, recalling the group $E$ discussed in 2.7 and the fact that $W \cong \pi^{-1}(P_1)$, $W' \cong \pi^{-1}(P_2)$, we have a very similar configuration in $E$. However, the two configurations are not isomorphic and here the deviation between the cohomology of $E$ and that of $M_{12}$ is explained by (2.2). Indeed, applying (2.2) we have

\begin{equation}
H^*(E) \cong H^*(M_{12}) \otimes (H^*(V_3) \otimes \text{St}(GL_3(\mathbb{F}_2)))^{GL_3(\mathbb{F}_2)}.
\end{equation}

On the other hand, using [M] we can give an independent evaluation of $H^*(E)$ using the fact that $E$ appears as the normalizer of one of the two $(\mathbb{Z}/2)^3$ tori in $G_2(\mathbb{F}_q)$ for $q \equiv 5 \mod (8)$, [FM]. In particular the result is

\[ P_E(t) = \frac{q(t)}{(1-t^4)(1-t^5)(1-t^7)} \]

where $q(t) = 1 + x^2 + 3x^3 + 2x^4 + 4x^5 + 5x^6 + 4x^7 + 5x^8 + 4x^9 + 2x^{10} + 3x^{11} + x^{12} + x^{13}$. Note that $E$ is also a subgroup of the compact 14 dimensional Lie group $G_2$ and $q(t)$ is the Poincaré series of the compact 14 dimensional manifold $G/E$, [M]. It follows that the error term $(H^*(V_3) \otimes \text{St}(GL_3(\mathbb{F}_2)))^{GL_3(\mathbb{F}_2)}$ has Poincaré series

\[ e_3(t) = \frac{t^4 + t^5 + t^6 + 2t^7 + t^8 + t^9 + t^{10}}{(1-t^4)(1-t^5)(1-t^7)}, \]

a result which is useful for understanding the group $O'N$.

Example (4.4). The O'Nan group $O'N$ has order $460,815,505,920 = 2^9 3^4 5^7 11 19 31$, and in [AM2] we determine its poset space, obtaining the following picture.
From this picture some easy cancellations give

\[ H^*(O'N) \oplus H^*((\mathbb{Z}/4)^3 \cdot \mathcal{S}_4) \cong H^*((\mathbb{Z}/4)^3 \cdot \text{GL}_3(\mathbb{F}_2)) \oplus H^*((\mathbb{Z}/4 \cdot \text{SL}_3(\mathbb{F}_4) \times \mathbb{Z}/2), \]

and using (2.2) this can be reduced to

\[
H^*(O'N) \oplus H^*(S)^{\mathbb{Z}/2} \cong H^*(S)^{\mathbb{Z}/2} \oplus (H^*(V_3) \otimes \text{St})^{\text{GL}_3(\mathbb{F}_2)} \\
\oplus H^*(\mathbb{Z}/4 \cdot \text{SL}_3(\mathbb{F}_4) \times \mathbb{Z}/2)
\]

where \( S \cong (\mathbb{Z}/4)^3 \cdot (\mathbb{Z}/2)^2 \). The terms involving \( S \) are reasonably direct to evaluate while the term involving the Steinberg module is discussed in (4.2). The final term is essentially determined in [AM1]. The \( A_5 \) invariants discussed in §3 play a key part in the work there.

**Example (4.5).** For the third Mathieu group \( M_{22} \) we use a sporadic geometry described in [RSY] which also satisfies the hypotheses necessary for (2.1) to remain valid. The associated complex has the form
We apply (2.1), (2.2), to this picture and, after some cancellation we have
\[ H^*(M_{22}) \oplus H^*(Syl_2(M_{22})) \cong H^*(V_4 \times \Sigma_4) \oplus H^*(V_4 \times \Sigma_5) \]
\[ \oplus (H^*(V_3) \otimes St)^{GL_3(F_2)} \oplus (H^*(V_4) \otimes St)^{A_6}. \]

From this, the results in [AM3], and the discussion of invariants in §3 we obtain the Poincaré series for \( M_{22} \). The formula is very messy and not too illuminating, so we defer details to a further paper.

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