**MATH 528: Homework III**

*Due Date: Thursday, April 3*

1. Let $f : \xi \to \eta$ denote a map of vector bundles. Show that the following are equivalent:

   (a) im $f$ is a subbundle
   (b) ker $f$ is a subbundle
   (c) the dimensions of the fibers of im $f$ are locally constant
   (d) the dimensions of the fibers of ker $f$ are locally constant.

2. A smooth closed manifold is said to be parallelizable if its tangent bundle can be trivialized. Show that $\mathbb{S}^3$ is parallelizable but that $\mathbb{S}^2$ is not. Use this to construct an example of a module $P$ over the ring $\Lambda = \mathbb{R}[x_0, x_1, x_2]/(x_0^2 + x_1^2 + x_2^2 - 1)$ which satisfies (i) $P \oplus \Lambda \cong \Lambda^3$; and (ii) $P$ is not free as a $\Lambda$–module. [This is an example of a *stably free* module which is not free].

3. Let $1 \to K \to G \to N \to 1$ denote an extension of finite groups. If $K$ and $N$ are perfect, show that $G$ is also perfect, and prove the existence of a five-term exact sequence

$$0 \to H^3(BN, \mathbb{Z}) \to H^3(BG, \mathbb{Z}) \to H^3(BK, \mathbb{Z})^N \to H^4(BN, \mathbb{Z}) \to H^4(BG, \mathbb{Z}).$$

4. Calculate the cohomology rings $H^*(\Omega\mathbb{S}^5, \mathbb{Z})$ and $H^*(\Omega SU(3), \mathbb{Z})$.

5. Calculate the cohomology ring $H^*(BSO(3), \mathbb{F}_2)$.

6. Construct a map $f : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{C}P^\infty$ which induces multiplication by an integer $n > 0$ on the second homology group. Let $F_n$ denote the fiber of this map. Compute the cohomology rings $H^*(F_n, \mathbb{Q})$, $H^*(F_n, \mathbb{F}_p)$ and $H^*(F_n, \mathbb{Z})$, where $p$ is a prime number that divides $n$.

7. (*Hurewicz Theorem*) Let $X$ be a connected space with $\pi_i(X) = 0$ for all $i < n$, where $n \geq 2$. Show that this implies that $H_i(X, \mathbb{Z}) = 0$ for all $i < n$ and that $\pi_n(X) \cong H_n(X, \mathbb{Z})$. [Hint: assume the fact that $H_1(X, \mathbb{Z})$ is the abelianization of $\pi_1(X)$ and use the Serre spectral sequence for the path fibration $\Omega(X) \to PX \to X$].

8. Let $X$ be a CW complex such that $H^*(X, \mathbb{F}_2)$ is generated by an element $x$ in degree $d$ such that $x^2 \neq 0$. Prove that $d$ is a power of two.