Last time we talked about convergence/divergence of certain integrals.

Example: \[ \int_0^1 \frac{1}{x^p} \, dx \text{ converges iff } p < 1 \text{ if and only if} \]

where \( p \) can be any rational number.

Today, we'll consider functions with an interior singularity.

\[ I = \int_{-1}^1 \frac{1}{x^2} \, dx \]
I = \int_{-1}^{0} \frac{1}{x^2} \, dx + \int_{0}^{1} \frac{1}{x^2} \, dx

\text{\uparrow}\quad \text{\uparrow}
\text{diverges}\quad \text{diverges}

= \text{does not exist}

\text{Integral diverges / \infty / DNE.}

\int_{-1}^{1} \frac{1}{x^2} \, dx \quad \text{diverges.}

Ex.: \quad I = \int_{0}^{2} \frac{1}{(x-1)^{\frac{2}{3}}} \, dx

Since integrand \frac{1}{(x-1)^{\frac{2}{3}}} has a singularity at x = 1 we break down the interval
\[ [0, 3] = [0, 1) \cup (1, 3] \]

\[
I = \int_0^1 \frac{1}{(x-1)^{2/3}} \, dx + \int_1^3 \frac{1}{(x-1)^{2/3}} \, dx
\]

\[
= \lim_{a \to 1^-} \int_0^a \frac{1}{(x-1)^{2/3}} \, dx + \lim_{b \to 1^+} \int_b^3 \frac{1}{(x-1)^{2/3}} \, dx
\]

\[
= \lim_{a \to 1^-} \frac{(x-1)^{1-2/3}}{1-2/3} \bigg|_0^a + \lim_{b \to 1^+} \frac{(x-1)^{1-2/3}}{1-2/3} \bigg|_b^3
\]

\[
= \lim_{a \to 1^-} \frac{3}{1} \left[ (x-1)^{1/3} \right]_0^a + \lim_{b \to 1^+} 3 \left[ (x-1)^{1/3} \right]_b^3
\]

\[
= \lim_{a \to 1^-} 3 \left[ (a-1)^{1/3} - (-1)^{1/3} \right] + \lim_{b \to 1^+} 3 \left[ (3-1)^{1/3} - (b-1)^{1/3} \right]
\]
\[
\lim_{a \to 1^-} 3 \left[ (a-1)^{\frac{1}{3}} - (-1)^{\frac{1}{3}} \right] \\
+ \\
\lim_{b \to 1^+} 3 \left[ 2^{\frac{1}{3}} - (b-1)^{\frac{1}{3}} \right] \quad \text{Plugging in } a = 1, b = 1 \\
= 3 \left[ (1-1)^{\frac{1}{3}} + 1 \right] + \\
3 \left[ 2^{\frac{1}{3}} - (1-1)^{\frac{1}{3}} \right] \\
= 3 \left[ 0 + 1 \right] + 3 \left[ 2^{\frac{1}{3}} - 0 \right] \\
= 3 + 3 \cdot 2^{\frac{1}{3}} = 3 (1 + 2^{\frac{1}{3}})
\]

Example: \( I = \int_{1}^{2} \frac{x}{\sqrt{x^2 - 1}} \, dx \)

Is it convergent or divergent?

\[
I = \lim_{a \to 1^+} \int_{a}^{2} \frac{x}{\sqrt{x^2 - 1}} \, dx
\]
\[ \int_{\frac{a}{2}}^{\frac{a}{2}} \frac{x}{a \sqrt{x^2 - 1}} \, dx = \int_{\frac{a}{2}}^{\frac{a}{2}} \frac{2x}{a \sqrt{x^2 - 1}} \, dx \]

\[ x^2 - 1 = u \]

\[ 2x = u' \]

\[ = \frac{1}{2} \int_{\frac{a}{2}}^{\frac{a}{2}} \frac{1}{\sqrt{u}} \, du \]

\[ = \frac{1}{2} \left( \frac{1}{u} \right) \bigg|_{\frac{a^2 - 1}{u^2}}^{\frac{a^2 - 1}{u^2}} \]

\[ = \sqrt{u} \bigg|_{\frac{a^2 - 1}{u^2}}^{\frac{a^2 - 1}{u^2}} \]

\[ = \sqrt{3} - \sqrt{a^2 - 1} \]

\[ I = \lim_{a \to 1^+} \left( \sqrt{3} - \sqrt{a^2 - 1} \right) = \sqrt{3} - \sqrt{0} = \sqrt{3} \]

\[ \int_{a}^{b} f(u(x)) u'(x) \, dx = \int_{u(a)}^{u(b)} f(u) \, du \]
Work

Work is the energy expended acting against a force. For example, energy expended in moving a weight against gravity.

Notations:

- \( t \) - time (seconds)
- \( m \) - mass (Kg)
- \( S \) - position (metres)

Newton’s 2nd Law
Force = mass \times \text{acceleration} \\[ F = m \times \frac{d^2s}{dt^2} \] \quad \text{(kg m/s^2)} \quad \text{aka Newton (N)}

Thus, work done against constant force $F$ when the object moves a distance $d$, is $F \times d$ \\[ W = F \cdot d \] \quad \text{(N \cdot m)} \quad \text{aka Joule (J)}

Example: How much work is done in moving a 1 Kg book from floor to a shelf?
height 2 metres.

\[ W = F \cdot d \]
\[ d = 2 \text{m} \]
\[ F = 1 \text{kg} \times 9.8 \text{ m/s}^2 \]
\[ = 9.8 \text{ N} \]

\[ W = 9.8 \text{ N} \times 2 \text{m} = 19.6 \text{ J} \]

If the force \( F \) varies according to distance, the work done as \( W = \int_a^b F(x) \, dx \)

**Hooke’s law** \( F = k \cdot x \)

\( x = \text{amount of stretching} \), \( k = \text{spring constant} \)
Example: A spring has a natural length of 20 cm. If a 25 N force is required to keep it stretched at a length of 30 cm, how much work is required to stretch it from 20 cm to 25 cm?

\[ F = kx \]
\[ 25N = k (0.30 - 0.20) \]
\[ 25N = k (0.1) \]
\[ k = \frac{25}{0.1} = 250 \text{ N/m} \]

\[ W = \int_0^{0.05} F(x) \, dx = \int_0^{0.05} 250x \, dx \]

\[ W = \frac{1}{2} \times 250 \times 0.05^2 \]

\[ W = 6.25 \text{ J} \]
\[ = 250 \times \frac{x^2}{2} \bigg|_0^w \]
\[ = 250 \left( \frac{(0.05)^2}{2} \right) \]
\[ = 125 \times \left( \frac{5}{100} \right)^2 \]
\[ = 125 \times \frac{25}{10^4} = 0.3125 \text{J} \]

Example: A leaky bucket weighing \(5\text{N}\) is lifted 20 m into the air at constant speed. The bucket starts with 2 m of water and leaks at a constant rate. It finishes to drip just as it reaches the top.
How much work was done lifting the water alone?