On the number of generators of an étale algebra

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Notation

\( R \) is a commutative Noetherian ring containing an infinite field \( k \).

We begin the classical Forster-Swan theorem.

Theorem

Let \( R \) be a ring. Let \( M \) be a finitely \( R \)-generated module. Suppose for each maximal ideal \( m \), \( M \otimes_R R/m \) can be generated by \( n \) elements as an \( R/m \)-module. Then \( M \) can be generated by \( n + \dim R \) elements.
This theorem was vastly generalized by Uriya First and Zinovy Reichstein.

**Theorem**

Let $A$ be an arbitrary $R$-algebra. Suppose $A$ is a finite $R$-module. Finally, suppose for each maximal ideal $m$, $A \otimes_R R/m$ can be generated by $n$ elements as an $R/m$-algebra. Then $A$ can be generated by $n + \dim R$ elements.

1. If $A$ is an algebra with trivial multiplication, recover Forster-Swan theorem.
2. If $A$ is an étale algebra over $R$, then $A$ can be generated by $1 + \dim R$ elements.
3. If $A$ is an Azumaya algebra over $R$, then $A$ can be generated by $2 + \dim R$ elements.
**Question**: Can we improve the bounds?

**Answer**: Not for modules.

Richard Swan produced examples of projective modules for which the upper bound is sharp.
Briefly, finite étale algebras are generalization of separable (field) extensions.

Standard Example: $R[x]/(f)$ over $R$ where $f \in R[x]$ is a separable polynomial.

On an scheme $X$ a finite étale algebra is a scheme $Y$ and finite morphism $\pi : Y \to X$ such that there exists an affine open cover $X = \bigcup_i U_i$ such that $\mathcal{O}_Y(\pi^{-1}(U_i)) \cong \mathcal{O}_X(U_i)[X]/(f)$ where $f$ is a separable polynomial.

Topologically, finite étale algebras are analogs of finite covering maps.

Diagram:

```
\begin{tikzpicture}
  \node (U) at (0,0) {$U$};
  \node (p) at (0,-1) {$p$};
  \node (p^{-1}(U)) at (0,-2) {$p^{-1}(U)$};
  \draw[->] (U) -- (p);
  \draw[->] (p^{-1}(U)) -- (U);
\end{tikzpicture}
```

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Is this upper bound optimal for étale algebras?

**Theorem (\_, Ben Williams)**

For each integer $n \geq 1$ there exist a ring $R^{(n)}$ of dimension $n$ and a finite étale algebra $A^{(n)}$ over $R^{(n)}$ that cannot be generated by fewer than $n + 1$ elements.
Idea of Proof I

Fix a degree $t$.

Construct a “universal” variety $B^n_t$ which classifies étale algebras (of degree $t$) having a generating set of $n$ elements.

$$
\begin{array}{c}
Y \xleftarrow{\text{closed}} \mathbb{A}^n \\
\downarrow \\
\downarrow \\
X \\
\end{array} 
\quad \xrightarrow{\sim} \quad \text{Hom}(X, B^n_t)
$$

- Morphism to $B^n_t$ is independent of the choice of generators.
- There exists natural maps $i_n : B^{n-1}_t \rightarrow B^n_t$.
- The maps $X \rightarrow B^n_t \rightarrow B^N_t$ are $\mathbb{A}^1$-homotopic for some $N >> n$. 
Then for a suitable $X$ there exists a morphism $X \to B_t^n$ such that it does not “factor” through $B_t^{n-1}$.

Obtain obstruction to generation by applying homotopy invariant functors. They include motivic cohomology or singular cohomology (if the base field is $\mathbb{R}$).
Thank you for listening!
Consider the $C_2 = \langle \sigma \rangle$-action on \( \frac{\mathbb{R}[x, y]}{(x^2 + y^2 - 1)} \) given by
\[
\sigma x = -x, \quad \sigma y = -y.
\]

Then $A$ cannot be generated by 1 element over $R$. 
Remark

Clearly $A$ can be generated by $(x, -x)$ and $(y, -y)$.

1. If $\zeta = (q(x, y), q(-x, -y))$ generated $A$ then it also generates the trivial étale algebra over $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$.

2. In that case $q(x, y) - q(-x, -y)$ is a unit in $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$.

3. But then $q(x, y) - q(-x, -y) = \lambda \in \mathbb{R}^*$.

4. All homogeneous components of $q(x, y)$ have even degree. Consequently, $q(x, y) = q(-x, -y)$.

5. But then $(q(x, y), q(x, y))$ cannot generate the trivial étale algebra.
**Question**
Can one produce examples over $\mathbb{C}$.

**Remark**
We can produce examples which can’t be generated by $n/2$ elements where $n = \dim R$. 