MATH 100 Section 108 – 2019W
Notes on Logarithms

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1   Definition

Fix a number $q > 0$, $q \neq 1$ which will be our base. We saw in the lectures that the exponential function q^x is one-to-one (Because it is strictly increasing if $q > 1$, and strictly decreasing if $0 < q < 1$.); therefore, it has an inverse. We defined the inverse of this functions as

$$\log_q(x).$$

In other words,

$$y = \log_q(x) \qquad \text{if and only if} \qquad q^y = x.$$

So, we also have

$$q^{\log_q(x)} = x, \qquad \text{and} \qquad \log_q(q^x) = x.$$

Note that since $q > 0$, for any value of $x \in \mathbb{R}$, we have $q^x > 0$. Moreover, if $q > 1$; then $\lim_{x \to -\infty} q^x = 0$, and $\lim_{x \to \infty} q^x = +\infty$. Similarly, if $0 < q < 1$; then $\lim_{x \to -\infty} q^x = +\infty$, and $\lim_{x \to \infty} q^x = 0$. So, the range of values q^x can take is in the interval $(0, \infty)$, and q^x attains any such value. Thus, the domain of the function $\log_q(x)$ is $(0, \infty)$.

We had a special, natural base, which is called the *Euler's number*, denoted by e. In our course (and in nearly all of the math courses), when we simply write $\log(x)$, we mean the logarithm in this natural base, i.e., for us

$$\log(x) = \log_e(x) = \ln(x).$$

However, more often than not we mention at the beginning which one we are using to avoid ambiguity. One can use any of these to mean logarithm in natural base as long as they are clear, and mention what they mean.

The graphs of $y = e^x$ and $y = \log(x)$ are given in Figure 1. The blue curve is $y = e^x$, the green one is $y = \log(x) = \ln(x)$, and the dashed orange curve is the line $y = x$. Pay special attention to the domain and the range of each function. Note that these functions are inverses of each other, so their graphs are reflections of one another with respect to the line $y = x$.
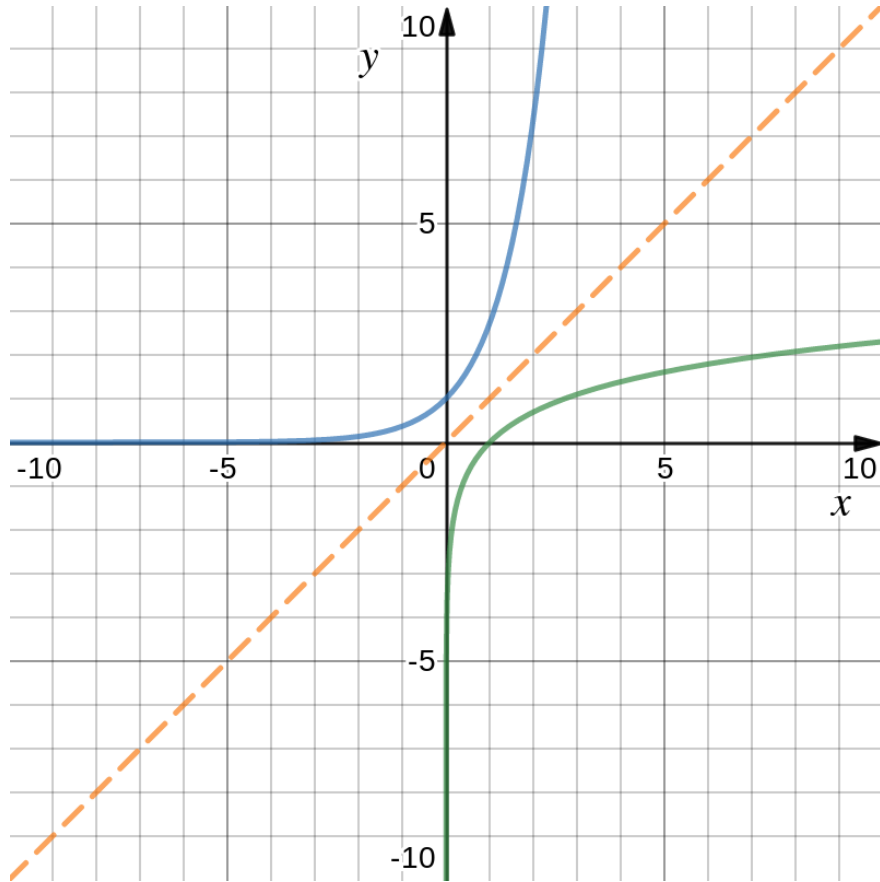
Figure 1: The curves $y = e^x$ (in blue), $y = \log(x)$ (in green), and the line $y = x$ (in orange)

1.1 Examples

Problem 1. Simplify $\log_{10}\left(100^4\right)$.

Solution. Since $100^4 = (10^2)^4 = 10^8$, and $\log_{10}(x)$ is the inverse function of 10^x, we get

$$\log_{10}\left(100^4\right) = \log_{10}\left(10^8\right) = 8.$$ ■

Problem 2. Simplify $\log_2(0.125)$.

Solution. Since $0.125 = 1/8 = (1/2)^3 = 2^{-3}$, and $\log_2(x)$ is the inverse function of 2^x, we get

$$\log_2(0.125) = \log_2\left(2^{-3}\right) = -3.$$ ■

2 Properties of the Logarithm

Let $q > 0$, $q \neq 1$ be a fixed base. Logarithms satisfy the following algebraic rules.

(a) $\log_q(A \times B) = \log_q(A) + \log_q(B)$ for any $A, B > 0$.

(b) $\log_q(A/B) = \log_q(A) - \log_q(B)$ for any $A, B > 0$.

2

(c) $\log_q(A^r) = r \log_q(A)$ for any $A > 0$, and any real number $r \in \mathbb{R}$.

Remark. These are true in the particular case of $q = e$, or $\log_q(x) = \log(x) = \ln(x)$.

Proof.   (a) **Slick Way.** We have

$$\log_q(A \times B) = \log_q\left(\left[q^{\log_q(A)}\right] \times \left[q^{\log_q(B)}\right]\right) = \log_q\left(q^{\log_q(A)} \times q^{\log_q(B)}\right)$$
$$= \log_q\left(q^{\left[\log_q(A)+\log_q(B)\right]}\right)$$
$$= \log_q(A) + \log_q(B).$$

Straightforward Way. Say $a = \log_q(A)$ and $b = \log_q(B)$. Then one has $A = q^a$ and $B = q^b$, so $A \times B = q^a \times q^b = q^{(a+b)}$. Taking $\log_q$ of both sides, we get

$$\log_q(A \times B) = \log_q(q^{(a+b)}) = a + b = \log_q(A) + \log_q(B).$$

(b) **Slick Way.** We have

$$\log_q(A/B) = \log_q\left(\left[q^{\log_q(A)}\right] / \left[q^{\log_q(B)}\right]\right) = \log_q\left(q^{\log_q(A)}/q^{\log_q(B)}\right)$$
$$= \log_q\left(q^{\left[\log_q(A)-\log_q(B)\right]}\right)$$
$$= \log_q(A) - \log_q(B).$$

Straightforward Way. Say $a = \log_q(A)$ and $b = \log_q(B)$. Then one has $A = q^a$ and $B = q^b$, so $A/B = q^a/q^b = q^{(a-b)}$. Taking $\log_q$ of both sides, we get

$$\log_q(A/B) = \log_q(q^{(a-b)}) = a - b = \log_q(A) - \log_q(B).$$

(c) **Slick Way.** We have

$$\log_q(A^r) = \log_q\left(\left[q^{\log_q(A)}\right]^r\right) = \log_q\left(q^{\left[r \log_q(A)\right]}\right)$$
$$= r \log_q(A).$$

Straightforward Way. Say $a = \log_q(A)$. Then we have $A = q^a$, so $A^r = (q^a)^r = q^{(r \times a)}$. Taking $\log_q$ of both sides, we get

$$\log_q(A^r) = \log_q\left(q^{(r \times a)}\right) = r \times a = r \log_q(A). \qquad \square$$

2.1 Example

Problem 3. Let

$$f(x) = \log\left(\frac{10}{x^2}\right) + 2\log(x) - \log(10 + x)$$

be defined for all $x > 0$. Simplify $f(x)$ into a single logarithm.

Solution.

$$\begin{aligned}
f(x) &= \log\left(\frac{10}{x^2}\right) + 2\log(x) - \log(10 + x) \\
&= \log\left(\frac{10}{x^2}\right) + \log\left(x^2\right) - \log(10 + x) \\
&= \log\left(\frac{10}{x^2}x^2\right) - \log(10 + x) \\
&= \log\left(10\right) - \log(10 + x) \\
&= \log\left(\frac{10}{10 + x}\right).
\end{aligned}$$
∎

3 Changing the Base

For any two bases $q, b > 0$, $q, b \neq 1$, we have the following really useful base-change formula:

$$\log_q(x) = \frac{\log_b(x)}{\log_b(q)}$$

for any $x > 0$. In particular, we have

$$\log_q(x) = \frac{\log(x)}{\log(q)} = \frac{\ln(x)}{\ln(q)}.$$

Proof. Say $\log_q(x) = a$; hence, $x = q^a$. Taking $\log_b$ of both sides and using the Property (c) above, we get

$$\log_b(x) = \log_b\left(q^a\right) = a\log_b(q) = \log_q(x)\log_b(q).$$

Rearranging, we conclude

$$\log_q(x) = \frac{\log_b(x)}{\log_b(q)}.$$
□

3.1 Example

Problem 4. (a) Suppose your calculator can compute logarithms only in base 10. What would you enter to calculate $\ln(17)$?

(b) Suppose your calculator can compute only natural logarithms. What would you enter to calculate $\log_2(57)$?

(c) Suppose your calculator can compute logarithms only in base 2. What would you enter to calculate ln(2)?

Solution. (a)
$$\ln(17) = \log_e(17) = \frac{\log_{10}(17)}{\log_{10}(e)}.$$

(b)
$$\log_2(57) = \frac{\log_e(57)}{\log_e(2)} = \frac{\ln(57)}{\ln(2)}.$$

(c)
$$\ln(2) = \log_e(2) = \frac{\log_2(2)}{\log_2(e)} = \frac{1}{\log_2(e)}. \qquad \blacksquare$$
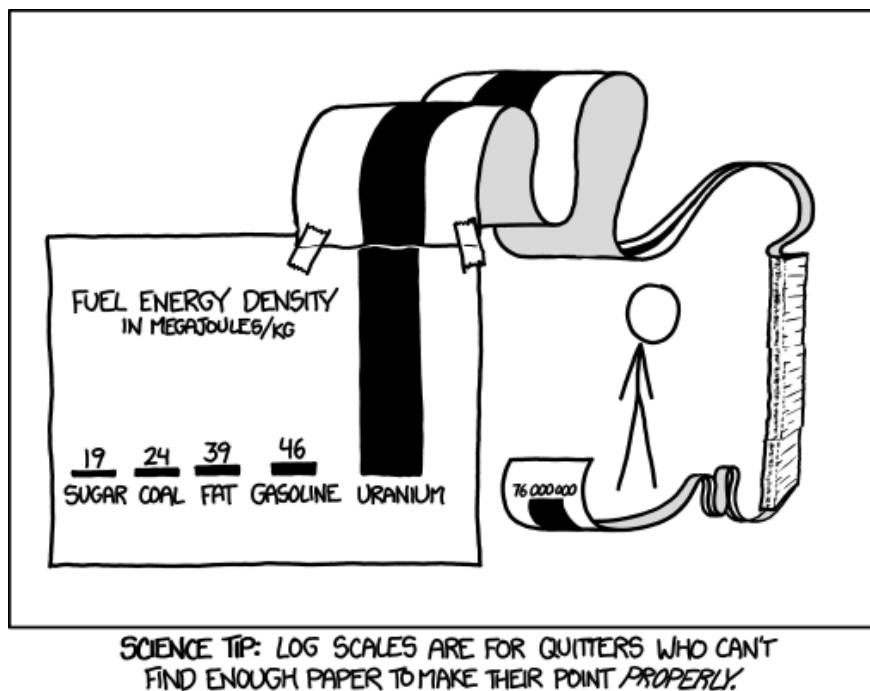
4 Misc.

Figure 2: `https://xkcd.com/1162/`

Ancient Proverb. Every graph looks linear in log-log scale.
(I have no idea where I've heard this from and I can't find it anywhere, but I know I've heard it from somewhere. —AAB)

(For mistakes, errors, typos, comments, and questions please email.)
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