MATH 100 Section 108 – 2019W
Practice Problems 2

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Problem 1. Where are the following functions continuous?

(a) **(Final 2016)** $f(x) = \dfrac{\sin(\pi x/2)}{\sqrt{1-x^2}}$

(b) **(Final 2015)** $f(x) = \dfrac{x^2-1}{\sqrt{x^2-x-6}}$

(c) **(I've literally just made this up. —AAB)** $f(x) = \dfrac{e^{\tan(x)}}{2+\cos(\sqrt{x})}$

(d) **(Hehehe. —AAB)** $f(x) = \dfrac{1}{\sqrt{1-[\tan(x)]^2}}$

Problem 2 (Final 2012). Let c be a constant, and define

$$f(x) = \begin{cases} x^2+c, & \text{if } x \le 1; \\ 2x-3, & \text{if } x > 1. \end{cases}$$

Find the value of c for which $f(x)$ is continuous everywhere.

Problem 3 (Final 2016). Let $f(x) = \dfrac{x}{x-2}$. Compute $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ using the definition of the derivative.

Problem 4. Using the definition of the derivative show that $f(x) = x|x|$ is differentiable at $x = 0$.

Problem 5. Using the definition of the derivative explain why $f(x) = \sqrt[3]{x}$ is not differentiable $x = 0$.
Hint: For any pair of real numbers a and b, we have the equality

$$a^3 - b^3 = (a-b)(a^2+ab+b^2).$$

Problem 6 (Final 2015). Is the function

$$f(x) = \begin{cases} \sqrt{1+x^2} - 1, & \text{if } x \le 0; \\ x^2 \cos(1/x), & \text{if } x > 0; \end{cases}$$

differentiable at $x = 0$? You must explain your answer using the definition of the derivative.

Problem 7. Find the values of the constants a and b such that the function

$$f(x) = \begin{cases} ae^x + \sin(x), & \text{if } x \le 0; \\ x^2 + 2x + b, & \text{if } x > 0; \end{cases}$$

is differentiable everywhere.

Problem 8. Differentiate the following functions.

(a) **(Final 2016)** $f(x) = x^2 e^x$

(b) **(Final 2016)** $f(x) = \dfrac{x^2+3}{2x-1}$

(c) **(Final 2015)** $f(x) = \dfrac{x^2}{x+1}$

(d) **(Final 2014)** $f(x) = e^x \cdot \cos(\pi x)$

Problem 9 (Final 2015). What is the equation of the tangent line to $f(x) = \sqrt{x}$ at the point $(4, 2)$?

Problem 10 (Final 2011). Find an equation of the tangent line to the curve $y = x^{3.5} - e^{3.5}$ at the point $(e, 0)$.

Problem 11 (Final 2014). Let $f(x)$ be a function differentiable at $x = 1$ and let $g(x) = f(x)/x^2$. The line tangent to the curve $y = f(x)$ at $x = 1$ has slope 3, while the line tangent to the curve $y = g(x)$ at $x = 1$ has slope 4. What is $f(1)$?

Problem 12 (Final 2016). Let $f(x)$ be a continuous function defined for all real numbers x. Suppose $f(x)$ is increasing on the intervals $(-\infty, -1)$ and $(3, \infty)$, decreasing on $(-1, 3)$; $f(-1) = 2$, and $f(3) = 1$. How many zeroes does $f(x)$ have?

(A) 0

(B) 1

(C) 2

(D) 3

(E) Cannot determine from the information given.

Problem 13 (Final 2012). Suppose the tangent line to the curve $y = f(x)$ at $x = 1$ passes through the points $(-2, 3)$ and $(0, 5)$. Find $f(1)$ and $f'(1)$.

Problem 14 (Final 2012). Find a function $f(x)$ such that $f'(x) = x^3$ and such that the line $x + y = 0$ is tangent to the graph of $y = f(x)$.

Problem 15 (Final 2016—Edited). Let $f(x)$ be a continuous function so that $|f(x) - \sin x| \leq 1/3$ for all x. Show that $f(x)$ has at least one zero in the open interval $(0, 2\pi)$.

Problem 16 (Final 2015—Edited). Show that the equation $2x^2 - 3 + \sin(x) + \cos(x) = 0$ has at least two solutions..

Problem 17 (Final 2011). If $y = f(x)$ is a continuous with domain $[0, 1]$ and range in $[3, 5]$. Show the line $y = 2x + 3$ intersects the graph of $y = f(x)$ at least once.
Note: Here, "with domain $[0, 1]$ and range in $[3, 5]$" means that for any $0 \leq x \leq 1$ we have $3 \leq f(x) \leq 5$. —AAB

Problem 18. Find the mistake in the following **WRONG** reasoning that reaches to a conclusion which is **FALSE**.

I. $\tan(\pi/4) = 1 > 0$.

II. $\tan(3\pi/4) = -1 < 0$.

III. The function $f(x) = \tan(x)$ is continuous on the closed interval $[\pi/4, 3\pi/4]$.

IV. So, by the IVT, there exists a point c in $(\pi/4, 3\pi/4)$ such that $\tan(c) = 0$.

Note: The function $\tan(x)$ is **NEVER** 0 in the open interval $(\pi/4, 3\pi/4)$. I'm asking you what went wrong in the reasoning.
Hint: Have a look at the graph of $\tan(x)$.

Problem 19 (Final 2012). Let $f(x) = g(2\sin x)$, where $g'(\sqrt{2}) = \sqrt{2}$. Find $f'(\pi/4)$.

Problem 20 (Final 2010. This problem is cool as f[Censored]! —AAB). Two points on the surface of the Earth are called *antipodal* if they are exactly at opposite points (with respect to the center of the Earth —AAB). For example, the North Pole and the South Pole are antipodal points. Prove that, at any given moment, there are two antipodal points on the equator with exactly the same temperature.
Hint: [Redacted. (Harm to Ongoing Matter.) I want to see if you can do it by yourselves. —AAB]

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