MATH 100 Section 108 – 2019W
In-Class Problem Sheet 2 — Solutions

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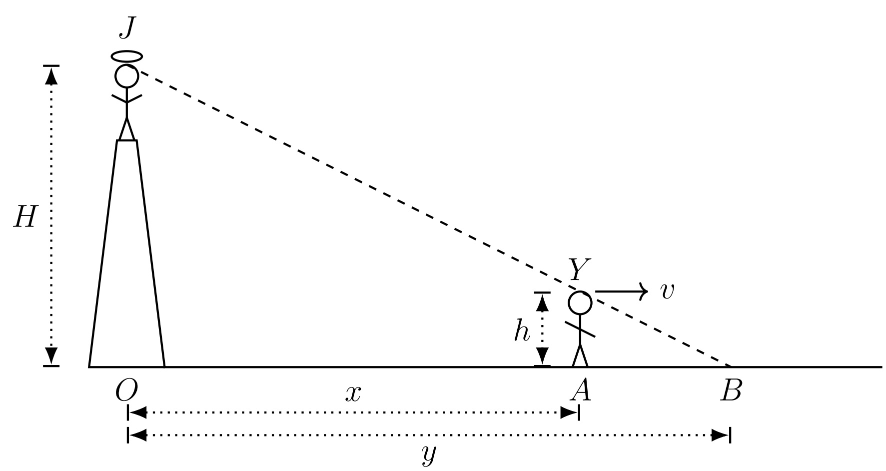
1 Related Rates Continued

Problem 1.1 (Jesus illuminating Yeezus, who is illuminating us). Suppose Jesus is on top of a hill so that He is 60 ft above from the ground. Kanye West, who is 6 ft, is on the ground, being illuminated with His divine knowledge. The horizontal distance between Jesus and Yeezus is 50 ft, and Yeezus is walking away from Jesus with a speed of 5 ft/s so that he can spread the Gospel.

We, the mortals on the ground, can gain this wisdom only after it reaches Ye's mind and gets converted into a gospel record such as *Jesus is King*. Assume that the divine knowledge behaves just like light (of course). Then how fast is the divine knowledge (or, the shadow of Ye's head) moving on the ground?

Solution. Consider the following figure describing the situation.



We are given the following: At the given time $H = 60$, $h = 6$, $v = \mathrm{d}x/\,\mathrm{d}t = 5$, and $x = 50$. We are asked to find $\mathrm{d}y/\,\mathrm{d}t$. So, we need to find an equation relating y to the information we're given.

Observe that the triangles BAY and BOJ are similar. So,

$$\frac{|AY|}{|OJ|} = \frac{|BA|}{|BO|};$$

hence,

$$\frac{h}{H} = \frac{y-x}{y}.$$

This gives that $hy = Hy - Hx$, or

$$y = \left(\frac{H}{H-h}\right) x.$$

This is the relation we're after. Differentiating both sides with respect to t, we get

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \left(\frac{H}{H-h}\right) \frac{\mathrm{d}x}{\mathrm{d}t}.$$

This is because H and h do not change with time, so they are constants with respect to t. So, plugging in $H = 60$, $h = 6$, and $\mathrm{d}x/\mathrm{d}t = 5$, we get

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \left(\frac{60}{60-6}\right) \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{10}{9}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{10}{9} \times 5 = \frac{50}{9} \text{ ft/s}.$$

Hallelujah!

Note: Observe that we haven't used the information that at the given time $x = 50$. Think about why. If you can't find a reasonable answer, ask me. ∎

2 Linear Approximation $[f(x) \approx L(x) = f(a)+f'(a)(x-a)]$

Problem 2.1 (A Real Story). When I, AAB, was a sophomore in undergrad, yours truly was taking a course on materials science and engineering. Being the complete idiot I am when it comes to the "real world," I of course forgot to bring my calculator to the midterm, and realized this only thirty seconds before the exam started.

One of the multiple-choice problems was about finding the ratio of electrons that manage to penetrate a slab of material, and I found the answer as

$$(0.998)^{10}.$$

However, all the choices were in decimals with two significant figures.

Realizing the deep mess I'm in and having no way out, I found the situation extremely funny. I decided to improvise, because I had nothing to lose but my pride. I knew Calculus, and I realized I could approximate functions. I thought linear approximation was the best I could do at that moment.

What did I do, and what was the correct answer I found?

Solution. I picked $f(x) = x^{10}$. I wanted to approximate $f(0.998)$. I also picked $a = 1$ since it's close to 0.998, and it's easy to calculate $f(a)$ and $f'(a)$ at that point.

I had $f(a) = f(1) = 1$. Also, since $f'(x) = 10x^9$, I had $f'(a) = f'(1) = 10$; hence the linear approximation,

$$f(x) \approx L(x) = f(a) + f'(a)(x-a),$$

became

$$f(x) \approx L(x) = 1 + 10(x - 1).$$

So, I found the answer as

$$(0.998)^{10} = f(0.998) \approx 1 + 10(0.998 - 1) = 1 + 10(-0.002) = 1 - 0.02 = 0.98,$$

which was the correct one. ■

Problem 2.2. Approximate $\sqrt{8.9}$ using a linear approximation.

Solution. We can pick $f(x) = \sqrt{x}$. We want to approximate $f(8.9)$. Let's use $a = 9$ since it's close to 8.9, it's easy to find $f(9) = \sqrt{9} = 3$, and it's easy to calculate

$$f'(a) = f'(9) = \left[\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x}\right]\bigg|_{x=9} = \left[\frac{1}{2\sqrt{x}}\right]\bigg|_{x=9} = \frac{1}{6}.$$

Then the linear approximation,

$$f(x) \approx L(x) = f(a) + f'(a)(x - a),$$

becomes

$$f(x) \approx L(x) = 3 + \frac{1}{6}(x - 9).$$

So,

$$\sqrt{8.9} = f(8.9) \approx 3 + \frac{1}{6}(8.9 - 9) = 3 - \frac{0.1}{6} = 3 - \frac{1}{60} = 2.98\overline{3}.$$ ■

Problem 2.3. Approximate $\sin(3)$ using a linear approximation. It's OK to have π in your answer.

Solution. We choose $f(x) = \sin(x)$. We want to approximate $f(3)$. Choose $a = \pi$ since it's close to 3, it's easy to find $f(a) = f(\pi) = \sin(\pi) = 0$, and it's easy to calculate

$$f'(a) = f'(\pi) = \left[\frac{\mathrm{d}}{\mathrm{d}x}\sin(x)\right]\bigg|_{x=\pi} = [\cos(x)]\big|_{x=\pi} = \cos(\pi) = -1.$$

Then the linear approximation,

$$f(x) \approx L(x) = f(a) + f'(a)(x - a),$$

becomes

$$f(x) \approx L(x) = 0 + (-1)(x - \pi) = \pi - x.$$

So,

$$\sin(3) = f(3) \approx \pi - 3.$$ ■

Problem 2.4. Approximate $e^{1/10}$ using a linear approximation.

Solution. Pick $f(x) = e^x$. We want to approximate $f(1/10)$. Let's choose $a = 0$ since it's close to $1/10$, it's easy to find $f(a) = f(0) = e^0 = 1$, and it's easy to calculate

$$f'(a) = f'(0) = \left[\frac{\mathrm{d}}{\mathrm{d}x}\, e^x\right]\Bigg|_{x=0} = \left[e^x\right]\big|_{x=0} = e^0 = 1.$$

Then the linear approximation,

$$f(x) \approx L(x) = f(a) + f'(a)(x - a),$$

becomes

$$f(x) \approx L(x) = 1 + 1(x - 0) = 1 + x.$$

So,

$$e^{1/10} = f(1/10) \approx 1 + \frac{1}{10} = 1.1. \qquad \blacksquare$$

3 Quadratic Approximation

$$\left[f(x) \approx P(x) = f(a) + f'(a)(x - a) + \frac{1}{2} f''(a)(x - a)^2 \right]$$

Problem 3.1. Approximate $\log(1.1)$ using a quadratic approximation.

Solution. We pick $f(x) = \log(x)$, and $a = 1$. We have $f(a) = f(1) = \log(1) = 0$. Moreover,

$$f'(a) = f'(1) = \left[\frac{\mathrm{d}}{\mathrm{d}x} \log(x)\right]\Bigg|_{x=1} = \left[\frac{1}{x}\right]\Bigg|_{x=1} = \frac{1}{1} = 1,$$

and

$$f''(a) = f''(1) = \left[\frac{\mathrm{d}}{\mathrm{d}x}\, \frac{1}{x}\right]\Bigg|_{x=1} = \left[-\frac{1}{x^2}\right]\Bigg|_{x=1} = -1.$$

Then the quadratic approximation,

$$f(x) \approx P(x) = f(a) + f'(a)(x - a) + \frac{1}{2} f''(a)(x - a)^2,$$

becomes

$$f(x) \approx P(x) = 0 + 1(x - 1) + \frac{1}{2}\left(-1\right)\left(x - 1\right)^2 = (x - 1) - \frac{1}{2}(x - 1)^2.$$

So,

$$\log(1.1) = f(1.1) \approx (1.1 - 1) - \frac{1}{2}(1.1 - 1)^2 = 0.1 - \frac{0.01}{2} = 0.1 - 0.005 = 0.095. \qquad \blacksquare$$

Problem 3.2. Approximate $\sqrt[3]{28}$ using a quadratic approximation.

Solution. We pick $f(x) = \sqrt[3]{x} = x^{1/3}$, and $a = 27$; hence, $f(a) = f(27) = \sqrt[3]{27} = 3$. Moreover,

$$f'(a) = f'(27) = \left[\frac{\mathrm{d}}{\mathrm{d}x}\, x^{1/3}\right]\bigg|_{x=27} = \left[\frac{1}{3}x^{-2/3}\right]\bigg|_{x=27} = \frac{1}{3}27^{-2/3} = \frac{1}{3}3^{-2} = \frac{1}{3^3}$$

and

$$f''(a) = f''(1) = \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{3}x^{-2/3}\right)\right]\bigg|_{x=27} = \left[\left(\frac{1}{3}\right)\left(\frac{-2}{3}x^{-5/3}\right)\right]\bigg|_{x=27} = \frac{-2}{3^2}27^{-5/3} = \frac{-2}{3^2}3^{-5} = \frac{-2}{3^7}.$$

Then the quadratic approximation,

$$f(x) \approx P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2,$$

becomes

$$f(x) \approx P(x) = 3 + \frac{1}{3^3}(x-27) + \frac{1}{2}\left(\frac{-2}{3^7}\right)(x-27)^2.$$

So,

$$\sqrt[3]{28} = f(28) \approx 3 + \frac{1}{3^3}(28-27) - \frac{1}{3^7}(28-27)^2 = 3 + \frac{1}{3^3} - \frac{1}{3^7}.$$ ■

Problem 3.3. Determine what $f(x)$ and a should be used so that you can approximate the following using a quadratic approximation.

(a) $\log(0.9)$

(b) $e^{-1/30}$

(c) $\sqrt[5]{30}$

(d) $(2.01)^6$

Solution. (a) $f(x) = \log(x)$ and $a = 1$.

(b) $f(x) = e^x$ and $a = 0$.

(c) $f(x) = \sqrt[5]{x}$ and $a = 32 = 2^5$.

(d) $f(x) = x^6$ and $a = 2$. ■