MATH 100 Section 108 – 2019W
In-Class Problem Sheet 1

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1 Exponential Decay/Growth $[f(t) = f(0)e^{kt}]$

Problem 1.1 (Radioactive Decay). A sample of radioactive matter is stored in a lab in 2000. In the year 2002, it is tested and found to contain 10 units of a particular radioactive isotope of Alperenium. In the year 2005, it is tested and found to contain only 2 units of that same isotope of Alperenium. How many units of the isotope of Alperenium were present in the year 2000?

Answer: $10\left(\sqrt[3]{5}\right)^2 \approx 29.$

Problem 1.2 (Population Dynamics). In 1963, the US Fish and Wildlife Service recorded a bald eagle population of 487 breeding pairs. In 1993, that number was 4015. How many breeding pairs would you expect there were in 2006? What about 2015?

Answer: In 2006, $487\left(\frac{4015}{487}\right)^{(43/30)} \approx 10016$; in 2015, $487\left(\frac{4015}{487}\right)^{(26/15)} \approx 18860.$

Problem 1.3 (Compound Interest, or *Cash Rules Everything Around Me.* C.R.E.A.M. *Get the money. Dolla' dolla' bill, y'all!*). Suppose you invest \$10,000 in to an account that accrues compound interest. After one month, your balance (with interest) is \$10,100. How much money will be in your account after a year?

Answer: $10,000 \times (1.01)^{12} \approx 11,268.25.$

Problem 1.4 (Radiocarbon Dating—The Real Deal). Researchers at Charlie Lake in BC have found evidence[1] of habitation dating back to around 8500 BCE. For instance, a butchered bison bone was radiocarbon dated to about 10,500 years ago. Suppose a comparable bone of a bison alive today contains 1 mg of ^{14}C. If the half-life of ^{14}C is about 5730 years, how much ^{14}C do you think the researchers found in the sample?

Answer: $2^{-10,500/5,730} \approx 0.28$ mg.

[1] http://pubs.aina.ucalgary.ca/arctic/Arctic49-3-265.pdf

2 Newton's Law of Cooling $[T(t) = [T(0) - A]e^{Kt} + A]$

Problem 2.1 (Safety First). A farrier forms a horseshoe heated to 400°C, then dunks it in a pool of room-temperature (25°C) water. The water near the horseshoe boils for 30 seconds, but the temperature of the pool as a whole hasn't changed appreciably. The horseshoe is safe for the horse when it's 40°C. When can the farrier put on the horseshoe?

Answer: 60 seconds after the farrier dunks the horseshoe into the pool.

Problem 2.2 (Five O'Clock Tea). A cup of just-boiled tea is put on a porch outside. After ten minutes, the tea is 40°C, and after 20 minutes, the tea is 25°C. What is the temperature outside?

Answer: 20°C.

Problem 2.3 (Murder on the Orient Express). Suppose a body is discovered at 3:45 pm, in a room held at 20°C, and the body's temperature is 27°C: not the normal 37°C. At 5:45 pm, the temperature of the body has dropped to 25.3°C. When did the owner of the body die?

Answer: 9:21 a.m.

3 Related Rates (Chain Rule *in Action*) $[\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}]$

Problem 3.1 (Toy Example: Q-T-π). Suppose P and Q are quantities that are changing over time, t. Suppose they are related by the equation

$$3P^2 = 2Q^2 + Q + 3.$$

If $\frac{\mathrm{d}P}{\mathrm{d}t} = 5$ when $P(t) = 1$ and $Q(t) = 0$, then what is $\frac{\mathrm{d}Q}{\mathrm{d}t}$ at that time?

Answer: 30.

Problem 3.2 ("Loglogloglog..."). A garden hose can pump out a cubic meter of water in about 20 minutes. Suppose you're filling up a rectangular backyard pool, 3 meters wide and 6 meters long. How fast is the water rising?

Answer: 1/360 meters per minute.

Problem 3.3 (Crossing the River Styx with Virgil). A cannonball is attached to a rope, which is attached to a pulley on a boat, at water level. The cannonball is taken 8 (horizontal) meters from its attachment point on the boat, then dropped in the water. The cannon ball sinks straight down, so its horizontal position doesn't change. The rope is being let out at a constant rate of one meter per second, and two seconds have passed. How fast is the ball descending?

$\boxed{\textbf{Answer: } 5/3 \text{ meters per second.}}$

Problem 3.4 (AAB, Filling up His Flask with Whiskey). You are pouring water into a large jar, through a funnel with an extremely small hole. The funnel lets water into the jar at 100 mL per second, and you are pouring water into the funnel at 300 mL per second. The funnel is shaped like a cone with height 20 cm and diameter at the top also is 20 cm. (Ignore the hole in the bottom.) How fast is the height of the water in the funnel rising when it is 10 cm high?

$\boxed{\textbf{Answer: } 8/\pi \text{ centimeters per second.}}$

Problem 3.5 (Getting Wet When It's Not Raining). A sprinkler is 3 meters away from a long, straight wall. The sprinkler sprays water in a circle, making three revolutions per minute. Let P be the point on the wall closest to the sprinkler. The water hits the wall at some spot, and that spot moves as the sprinkler rotates. When the spot where the water hits the wall is 1 m away from P, how fast is the spot moving?

$\boxed{\textbf{Answer: } 20\pi \text{ meters per minute.}}$

Problem 3.6 (Descartes's Emotional Roller Coaster). A roller coaster has a track shaped in part like the *folium of Descartes*: $x^3 + y^3 = 6xy$. (See Figure 1.) When it is at the position $(3, 3)$, its horizontal position is changing at 2 units per second in the negative direction. How fast is its vertical position changing?

$\boxed{\textbf{Answer: } \frac{\mathrm{d}y}{\mathrm{d}t} = 2, \text{ so 2 units per second in the positive direction.}}$

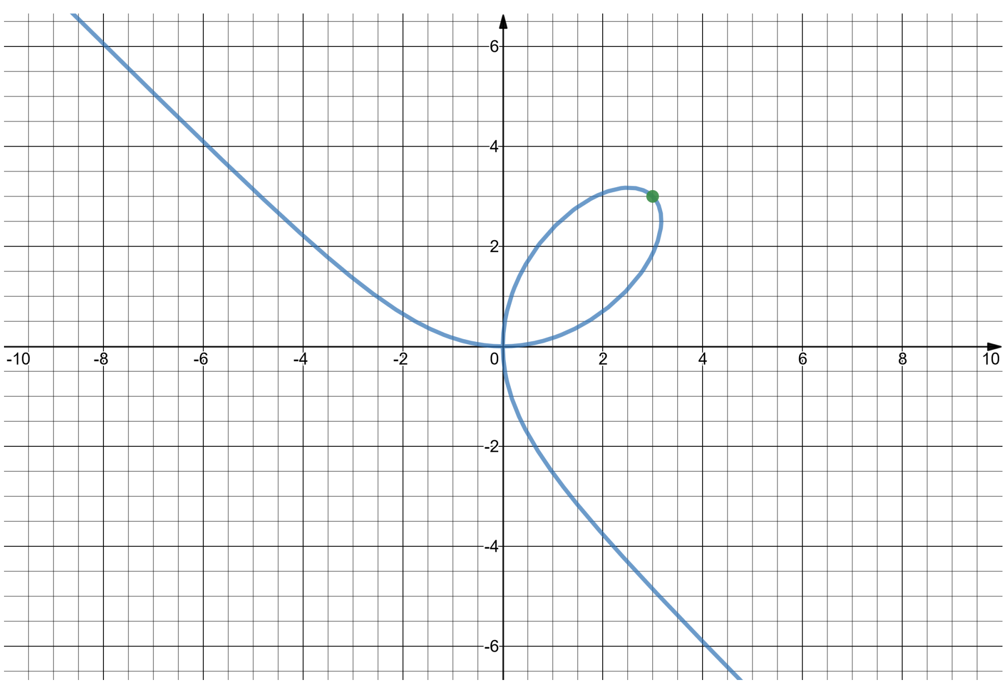

Figure 1: Folium of Descartes: $x^3 + y^3 = 6xy$

Problem 3.7 (No animals were harmed in the making of this film). Two dogs are tied with elastic leashes to a lamp post that is 2 meters from a straight road. At first, both dogs are on the road, at the closest part of the road to the lamp post. Then, they start running in opposite directions: one dog runs 3 meters per second, and the other runs 2 meters per second. After one second of running, how fast is the angle made by the two leashes increasing?

Answer: 25/26 radians per second.

Problem 3.8 (*"Imma let you finish..."*). Kanye West and Taylor Swift are attending at a gig at the same place. Eager not to cross paths, Taylor is one kilometer due east of the main stage, heading east, and Ye is two kilometers due north of the main stage, heading north (and to North, North West, his lovely and environmentally-conscious daughter). If Taylor is walking at 5 kph, and Ye is walking 7 kph, how fast is the distance between them increasing?

Answer: $19/\sqrt{5} \approx 8.50$ kph.

Problem 3.9 (#EuclidTime!). A triangle has one side that is 1cm long, and another side that is 2cm, and the third side is formed by an elastic band that can shrink and stretch. The two fixed sides are rotated so that the angle they form, θ, grows by 1.5 radians each second. Find the rate of change of the area inside the triangle when $\theta = \pi/4$.

Answer: $\dfrac{3}{2\sqrt{2}}$.