MATH 100 Section 108 – 2019W
Notes on the Definition of Derivative

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1 Derivatives

Geometric Interpretation

The geometric interpretation of the concept of derivative starts with what is called "the tangent line problem." Suppose we are given the graph of a function, $y = f(x)$, which is nice and smooth: without any corners or sharp points. Such a graph is shown in blue in Figure 1.

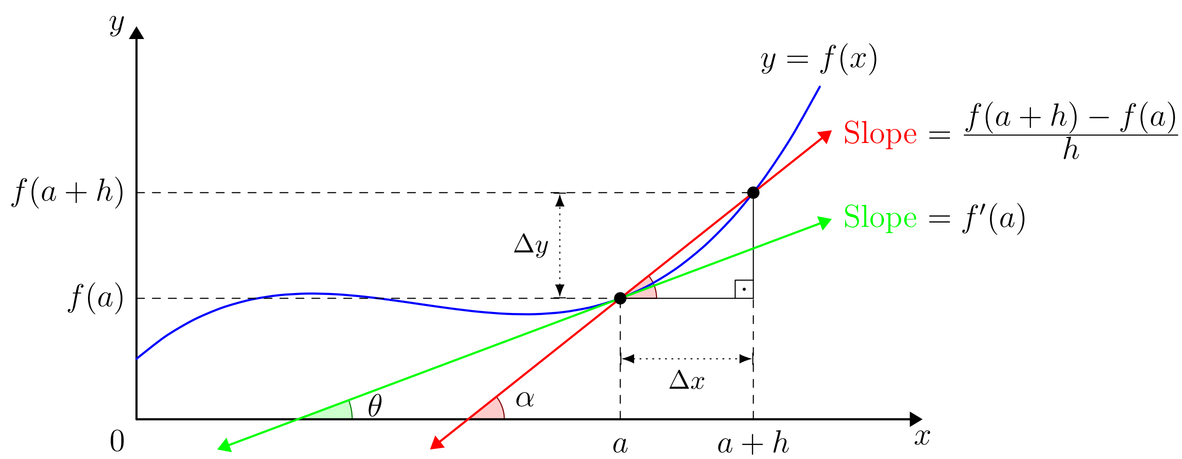

Figure 1: The Geometric Interpretation of the Derivative

What we want is to find the slope of the line that is tangent to our curve at the point where $x = a$.

The *tangent line* to a curve $y = f(x)$ at the point where $x = a$ is the line that "just touches" to the curve at that point. The tangent line to our blue curve at the point $x = a$ is shown in green in Figure 1. This can be understood through our physical intuition.

Suppose we are moving a point particle with a constant speed on our blue curve, and let it approach to the point $(a, f(a))$, from left to right. At the moment it reaches to the point where $x = a$, we let it go free. As Newton's first law of motion says, if on an inertial frame there is an object, on which there is no force acting, the object either remains still or moves on a line with a constant speed. So, our particle's path will follow the right half of the line shown in green. We do this also the other way around, and let our particle approach $(a, f(a))$ from right to left while it moves with constant speed on our blue curve. The moment it reaches to point where $x = a$ we let it go free. Intuitively, its path will follow the left half our green line. If our curve is nice enough; then this combined paths form a line, which is the tangent line: the line that "just touches" to the curve.

Another way to explain is this: if you zoom in enough on the graph of $y = f(x)$ around the point where $x = a$, the graph will look more and more like a straight line. *That line* is the tangent line. Among all the lines that pass from the point $(a, f(a))$, the tangent line is the one that best approximates (resembles) the original function around that point. What we want to find is the slope of this line.

Any line that is not vertical is the graph of a function of the form

$$y = mx + b,$$

where m and b are fixed constants. The constant m is called the *slope of the line.* The slope can be expressed in different ways as well. If (x_1, y_1) and (x_2, y_2) are two different points on a line, the slope is equal to

$$\frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Furthermore, the slope is equal to also

$$\tan\theta,$$

where θ is the angle that the line makes with the x-axis, measured in counterclockwise direction. Note that for a fixed line, all three of these are equal to each other.

Using only the point $(a, f(a))$, we cannot find the slope of the tangent line, our green line, because there are infinitely-many lines passing from a single point. We simply don't know how to choose the correct one! That's why we must "sneak up" on it, so we need to use limits.

First, we choose a point close to a, say $a + h$, where h is small (positive or negative), and draw the line connecting the two points on our blue curve: the points where $x = a$ and $x = a + h$. This line is called the *secant line*, and is shown in red in Figure 1.

To find the slope of this red line, we look at the right triangle—which has one side in red—formed in our figure. Using the length of the sides of this triangle, we get,

$$\text{Slope of the secant line} = \tan\alpha = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}.$$

Now, if we make h smaller and smaller, and let it approach to 0, we see that "the limit" of the slopes of these red lines should be equal to the slope of our tangent line, the line in green. (Just try to imagine it while moving the point where $x = a + h$ closer and closer to the point where $x = a$.) So, if the limit

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

exists; it is equal the slope of the tangent line. If that's the case, we say that $f(x)$ is differentiable at the point $x = a$, and denote this value by $f'(a)$.

In conclusion, the tangent line to a function at a point is the line which best "approximates" the function among all the lines that pass from the given point. The slope of that line is what we call the derivative. This is the geometric interpretation of the derivative.

Algebraic Definition

Definition 1. The ***derivative of*** $f(x)$ ***at a point*** $x = a$ is the slope of the tangent line to $f(x)$ passing from the point $(x, y) = (a, f(a))$. It is denoted by $f'(a)$. So,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h},$$

provided that this limit exists.

Alternative Definition. The same limit can be expressed also as

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a},$$

provided that this limit exists.

To see the equality between the two, substitute $x = a+h$ in the second one. Note that in this case, as h goes to 0, x goes to a, and vice versa. Depending on the function, sometimes using one is easier than using the other. Note that $f'(a)$ is not a function, it's a number.

If the limit above exists for $x = a$, we say that $f(x)$ is *differentiable at* a and denote it by either

$$f'(a) \qquad \text{or} \qquad \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=a}.$$

If the limit does not exist, then $f(x)$ is *not differentiable at* a.

Remark. If the function $f(x)$ is not defined at $x = a$, since "$f(a)$" does not exist (undefined), the limit above cannot exist as well. Hence, $f(x)$ cannot be differentiable at a point if it's not defined there.

Remark. We rarely use the definition of the derivative. Rather, we will mainly use some rules to calculate them after learning the derivatives of some common functions like polynomials, exponentials, trigonometric functions, etc.

Theorem. *If a function $f(x)$ is differentiable at a; then it is necessarily continuous at a.*

Example 1. Use the definition of the derivative to calculate the slope of the tangent line to $f(x) = x^2 - 5$ at $x = 3$. (First sketch the graph, and try to guess if $f'(3)$ should be positive, negative, or zero.)

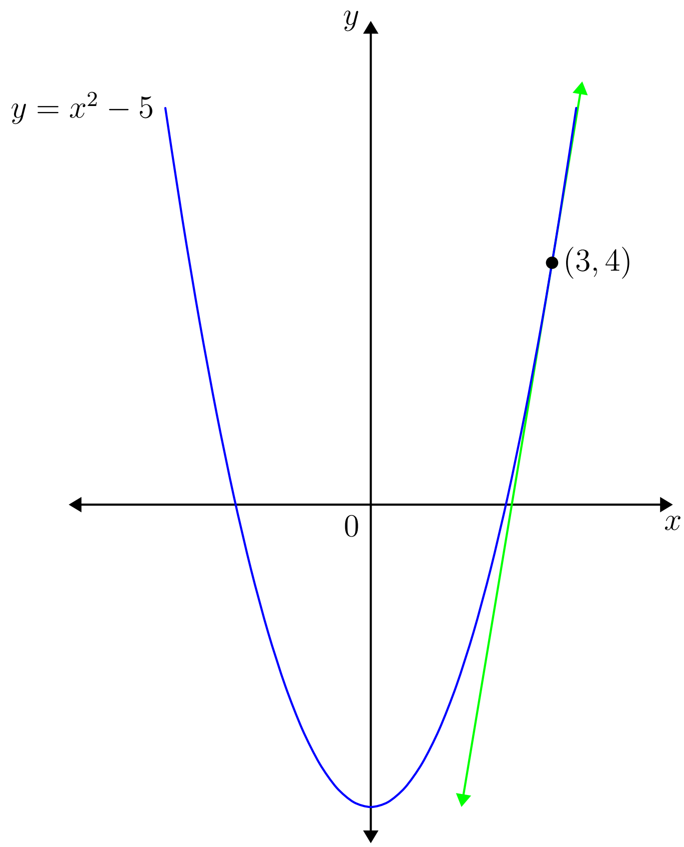

Figure 2: The Plot for $y = x^2 - 5$

Solution. Using the definition of the derivative, we get

$$\begin{aligned}
f'(3) &= \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} \\
&= \lim_{h \to 0} \frac{\left[(3+h)^2 - 5\right] - \left(3^2 - 5\right)}{h} \\
&= \lim_{h \to 0} \frac{\left[(3^2 + 6h + h^2) - 5\right] - (9 - 5)}{h} \\
&= \lim_{h \to 0} \frac{9 + 6h + h^2 - 5 - 4}{h} \\
&= \lim_{h \to 0} \frac{6h + h^2}{h} \\
&= \lim_{h \to 0} (6 + h) \\
&= 6.
\end{aligned}$$

This is plausible because the function $f(x) = x^2 - 5$ is increasing at $x = 3$, so the slope of the tangent line must be positive. ■

Derivative as a Function

Consider the above example again, where $f(x) = x^2 - 5$. Now let's try to calculate the derivative at an arbitrary point x. (In the particular case above, x will be 3.) Think of x as arbitrary but fixed. It's a placeholder until we substitute a number for it. Then, the same calculation will give

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\Big[(x+h)^2 - 5\Big] - \Big(x^2 - 5\Big)}{h} \\ &= \lim_{h\to 0} \frac{\Big[(x^2 + 2xh + h^2) - 5\Big] - \Big(x^2 - 5\Big)}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 5 - x^2 + 5}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h\to 0} (2x + h) \\ &= 2x. \end{aligned}$$

Note that in the last step, we think of x as fixed, and let h approach to 0.

So, we have calculated the derivative of $f(x) = x^2 - 5$ at any arbitrary point x, and we can conclude that $f'(x) = 2x$ for any real number x. Observe that this gives the same answer as above in the particular case where $x = 3$, namely $f'(3) = 2 \times 3 = 6$.

This motivates the following definition.

Definition 2. The ***derivative of*** $f(x)$ ***with respect to*** x is another function which is given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

provided that this limit exists.

If this limit exist for any x in an open interval, we say that *f(x) is differentiable in that interval.*

Here is some notation we will use for the derivative.

- $f'(a)$ (A number)
- $f'(x)$ (A function)
- $\displaystyle \frac{\mathrm{d}}{\mathrm{d}x} f(x)\Big|_{x=a}$ (A number)
- $\displaystyle \frac{\mathrm{d}}{\mathrm{d}x} f(x)$ (A function)

- $\dfrac{\mathrm{d}f}{\mathrm{d}x}(a)$ (A number)
- $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ (A function)

For mistakes, errors, typos, questions, and comments please email `aabulut@math.ubc.ca` —AAB

`derivative-defn-v1.pdf`: September 16th, 2019, 6:30 p.m. — Initial file

`derivative-defn-v2.pdf`: September 16th, 2019, 8:00 p.m. — Figure 2 and Theorem added, a few typos corrected

`derivative-defn-v2.1.pdf`: September 16th, 2019, 8:15 p.m. — Major typo in the definition corrected