MATH 100V01 – 2018W
Recitation Notes for Oct. 26th

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1 Derivatives

Definition 1. Let $f(x)$ be a function. The ***derivative of*** $f(x)$ is another function, usually denoted by $f'(x)$, defined by

$$f'(x) \coloneqq \lim_{h\to 0} \frac{f(x+h)-f(x)}{h},$$

wherever this limit exists.

If the limit above exists for $x = a$, we say that $f(x)$ is ***differentiable at*** a and denote it by either

$$f'(a) \qquad \text{or} \qquad \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=a}.$$

If the limit does not exist, f is ***not differentiable at*** a. The function f is said to be ***differentiable on an interval*** if it is differentiable at every point on that interval.

Remark. If the function f is not defined at $x = a$, since "$f(a)$" does not exist, the limit above cannot exist as well. Hence, f cannot be differentiable at a point if its not defined there.

Remark. We rarely use the definition of the derivative. Rather, we will mainly use some rules to calculate derivatives after learning the derivatives of some common functions like polynomials, exponentials, trigonometric functions, etc.

Geometric Interpretation

The geometric interpretation of the concept of derivative starts with what is called "the tangent line problem." Suppose we are given the graph of a function, $y = f(x)$, which is nice and smooth: without any corners or sharp points. Such a graph is shown in blue in Figure 1. What we want is to find the equation of the line that is tangent to our curve at the point $x = x_0$.

The *tangent line* to a curve $y = f(x)$ at the point $x = x_0$ is the line that "just touches" to the curve at that point. The tangent line to our blue curve at the point $x = x_0$ is shown in green in Figure 1. This can be understood through our physical intuition.

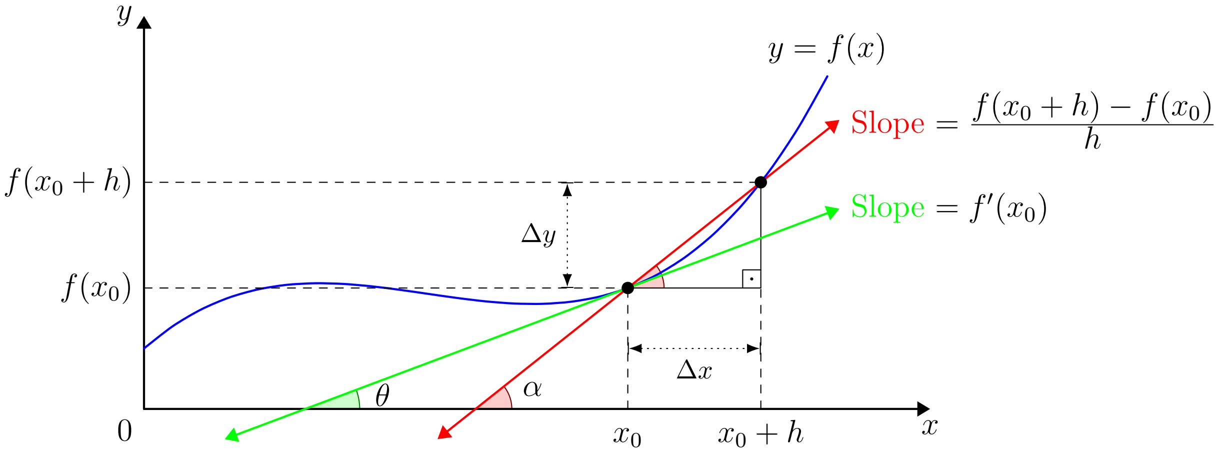

Figure 1: The Geometric Interpretation of the Derivative

Suppose we are moving a point particle with a constant speed on our blue curve. At the moment it reaches to the point $x = x_0$, from left to right, we let it go free. As Newton's first law of motion says: if on an inertial frame there is an object, on which there is no force acting, the object either remains still or moves on a line with a constant speed. So, our particle's path will follow the right half of the line shown in green. If we do this also the other way and let our particle, which is moving with constant speed on our blue curve, and let it go free when it reaches to point $x = x_0$ from right to left; then its path will follow the left half our green line. If our curve is nice enough; this combined paths form a line, which is the tangent line: the line that "just touches" to the curve. What we want to find is the equation of this line.

In the Euclidean plane, any line that is not vertical is actually the graph of a function of the form

$$y = mx + b,$$

where m and b are fixed constants. The constant m is called the *slope of the line*. Using the Pythagorean theorem, one can see that the value of m is equal to the tangent of the angle, say θ, at which the line crosses the x-axis.

We would like to find the value of m for our green line. To find it, we choose a point close to x_0, say $x_0 + h$, where h is small; and draw the line connecting the two points on our blue curve: the points where $x = x_0$ and $x = x_0 + h$. This line is called the *secant line*, and is shown in red in Figure 1.

To find the slope of this red line, we look at the right triangle—which has one side in red—formed in our figure. Using the length of the sides of this triangle, we get,

$$\tan\alpha = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{\Delta y}{\Delta x} = \frac{f(x_0+h) - f(x_0)}{h}.$$

Now, if we make h smaller and smaller, and let it approach to 0, we see that "the limit" of the slopes of these red lines should be equal to the slope of our tangent line, the line in green. Note that we couldn't have done this at the point $x = x_0$ only—and so needed a limit—because there are infinitely-many lines passing from a single point.

So, if this limit,

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h},$$

exists; we call that f is differentiable at the point $x = x_0$, and denote this value by $f'(x_0)$. The value of the derivative function $f'(x)$ gives us this limit wherever this limit exists, and is undefined otherwise.

In conclusion, the tangent line to a function at a point is the line which best "approximates" the function among all the lines that pass from the given point. The slope of that line is what we call the derivative. This is the geometric interpretation of the concept of derivative.

2 Problems

Problem 1 (Exercise 5). Let $f(x) = 2x + 3$. Confirm using the definition of derivative that $f'(x) = 2$. Why is this plausible?

Solution. Using the definition of the derivative, we get,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[2(x+h)+3]-[2x+3]}{h} \\ &= \lim_{h\to 0} \frac{2x+2h+3-2x-3}{h} = \lim_{h\to 0} \frac{2h}{h} = \lim_{h\to 0} 2 = 2, \end{aligned}$$

for any $x \in \mathbb{R}$.

This is plausible because the tangent line, at any point, to the *line* $y = 2x+3$ is nothing but the original line $y = 2x + 3$ itself, which has slope 2. ■

Problem 2 (Exercise 6). Let $f(x) = 1/x$. Calculate $f'(x)$ using the definition of derivative.

Solution. The definition of the derivative gives,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{1/(x+h)-1/x}{h} \\ &= \lim_{h\to 0} \frac{x/[x(x+h)]-(x+h)/[x(x+h)]}{h} = \lim_{h\to 0} \frac{x-(x+h)}{hx(x+h)} \\ &= \lim_{h\to 0} \frac{-h}{hx(x+h)} = \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x\times x} = -\frac{1}{x^2}, \end{aligned}$$

for any $x \neq 0$. If $x = 0$, since the function is not defined at $x = 0$, $f(x)$ cannot be differentiable at 0. ■

Problem 3 (Exercise 4). Sketch the graph, and come up with the expression for, a function that is continuous everywhere except at $x = 2$, and differentiable everywhere except at $x = 2$ and $x = 4$.

Solution. Here is a simple one:

$$f(x) = \begin{cases} 4, & \text{if } x < 4 \text{ and } x \neq 2; \\ x, & \text{if } x \geq 4. \end{cases}$$

The graph of f is given in Figure 2.

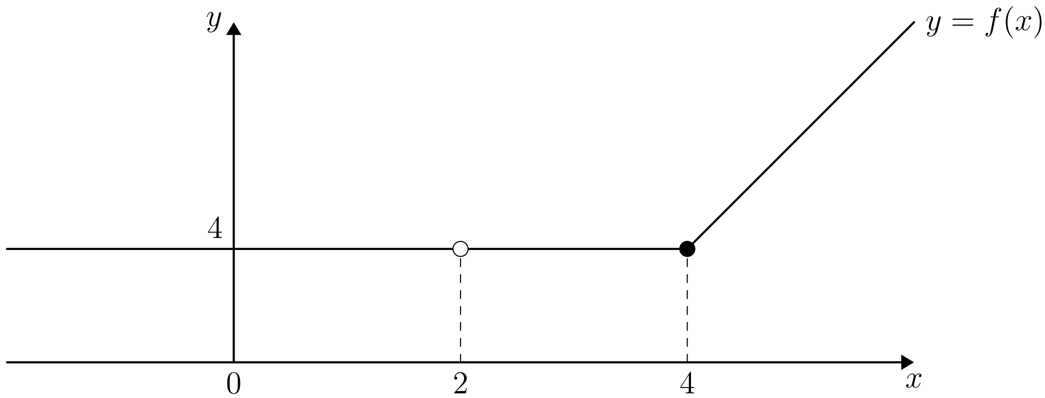

Figure 2: The Graph of $f(x)$

First, let us check continuity. If $x < 4$ and $x \neq 2$, we have $f(x) = 4$; and if $x > 0$, we have $f(x) = x$. These are both polynomials and every polynomial is continuous everywhere. So, $f(x)$ is continuous whenever x is neither 2 nor 4. Moreover, as can be seen from the graph,

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} 4 = 4,$$

and

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} x = 4,$$

So,

$$\lim_{x \to 4} f(x) = 4 = f(4).$$

This shows that f is continuous also at $x = 4$. Finally, since f is not defined at $x = 2$, it cannot be continuous there. We have shown that f is continuous everywhere except at $x = 2$.

Now let us check differentiability. Suppose $x < 4$ and $x \neq 2$; then for small enough h, we have $f(x + h) = 4$. Thus,

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{4 - 4}{h} = \lim_{h \to 0} 0 = 0.$$

So, if $x < 4$ and $x \neq 2$; the function f is differentiable at x with derivative 0.

Similarly, if $x > 4$; then for small enough h, we have $f(x + h) = x + h$. Hence,

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h) - x}{h} = \lim_{h \to 0} \frac{h}{h} = 1.$$

Therefore, if $x > 4$; the function f is differentiable at x with derivative 1.

However, if $x = 4$, we have,

$$\lim_{h\to 0^-} \frac{f(4+h)-f(4)}{h} = \lim_{h\to 0^-} \frac{4-4}{h} = \lim_{h\to 0^-} 0 = 0,$$

but

$$\lim_{h\to 0^+} \frac{f(4+h)-f(4)}{h} = \lim_{h\to 0^+} \frac{(4+h)-4}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1.$$

Since

$$\lim_{h\to 0^-} \frac{f(4+h)-f(4)}{h} \neq \lim_{h\to 0^+} \frac{f(4+h)-f(4)}{h},$$

the limit $\lim_{h\to 0}([f(4+h)-f(4)]/h)$ does not exist. So the function f is not differentiable at $x = 4$. On the other hand, f is not defined at $x = 2$, so it cannot be differentiable there.

In summary, we have shown that f is continuous everywhere except $x = 2$, and that f is differentiable everywhere except at $x = 2$ and $x = 4$. ■

3 Supplementary Problems

Problem 4 (Exercise 1). Write down definitions and sketch illustrations of the following statements.

(a) The function $f(x)$ is right-differentiable at $x = a$.

(b) The function $f(x)$ has a vertical tangent line at $x = a$.

Solution. (a) We say that the function $f(x)$ is ***right-differentiable at*** $x = a$ if the limit

$$\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$$

exists.

An example is the function given in the solution of Problem 3 (Exercise 4). Its graph can be seen on Figure 2. The function f is right-differentiable at $x = 4$ since, as done in the solution,

$$\lim_{h\to 0^+} \frac{f(4+h)-f(4)}{h} = \lim_{h\to 0^+} \frac{(4+h)-4}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1.$$

(b) We say that the function $f(x)$ has a ***vertical tangent line at*** $x = a$ if

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \pm\infty.$$

An example is the function $f(x) = \sqrt[3]{x-1} + 2 = (x-1)^{1/3} + 2$ at $x = 1$. Its graph can be seen on Figrue 3. (Note that $g(x) = \sqrt[3]{x}$ at $x = 0$ works just fine as well but the graph of f looks much better.)

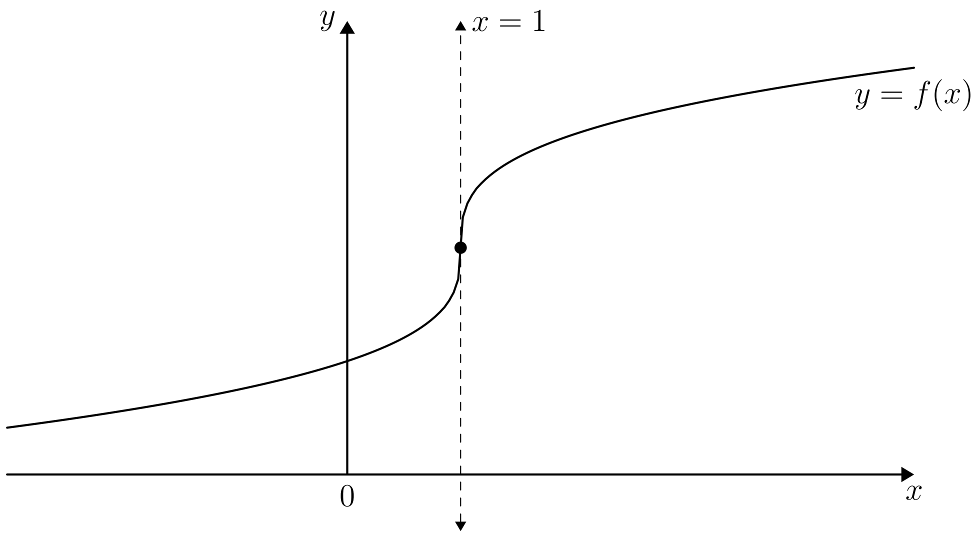

Figure 3: The Graph of $f(x)$

Observe that

$$\begin{aligned}\lim_{h\to 0}\frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}\\ &= \lim_{h\to 0}\frac{\left[((1+h)-1)^{1/3}+2\right]-\left[(1-1)^{1/3}+2\right]}{h}\\ &= \lim_{h\to 0}\frac{h^{1/3}}{h} = \lim_{h\to 0}\frac{1}{h^{2/3}} = +\infty.\end{aligned}$$

Remark. The function $f(x)$ is <u>not</u> differentiable at $x = 1$ since infinite limits do not exist.

■

Problem 5 (Exercise 1). Determine whether each of the following statements is true or false, and justify your answer.

(a) If $f(x)$ is differentiable; then $|f(x)|$ is differentiable.

(b) If $|f(x)|$ is differentiable; then $f(x)$ is differentiable.

(c) If $f(x)$ has a vertical asymptote $x = a$; then $f(x)$ is not differentiable at a.

Solution. (a) This statement is false. A counterexample is $f(x) = x$. The function f is differentiable everywhere since

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{x+h-x}{h} = \lim_{h\to 0}\frac{h}{h} = 1$$

for any $x \in \mathbb{R}$. However, $|f(x)| = |x|$ is not differentiable at $x = 0$. This is because the limit

$$\lim_{h\to 0}\frac{|f(0+h)|-|f(0)|}{h} = \lim_{h\to 0}\frac{|0+h|-|h|}{h} = \lim_{h\to 0}\frac{|h|}{h}$$

does not exist.

(b) This statement is false as well. A counterexample is the function

$$f(x) = \begin{cases} 1, & \text{if } x \neq 0; \\ -1, & \text{if } x = 0; \end{cases}$$

at $x = 0$. Since $|f(x)| = 1$ everywhere, $|f(x)|$ is differentiable everywhere. In particular $|f(x)|$ is differentiable at $x = 0$. However, since the limit

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{1 - (-1)}{h} = \lim_{h\to 0} \frac{2}{h}$$

does not exist; the function f is not differentiable at $x = 0$.

(c) This statement is true. This is because, as given in the solution of Problem 4 part (b), if f has a vertical asymptote at $x = a$; then,

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \pm\infty.$$

Since infinite limits do not exist, this limit does not exist; and so f is not differentiable at $x = a$.

■

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