Recitation Notes for Math 100:701 2018W1 Nov. 30th.

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1. Derivative

(1) Definition: Let $f(x)$ be a function. The derivative of $f(x)$ is another function, usually denoted by $f'(x)$, defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

where this limit exists.

If the limit above exists for a, we say that $f(x)$ is differentiable at a and denote it either

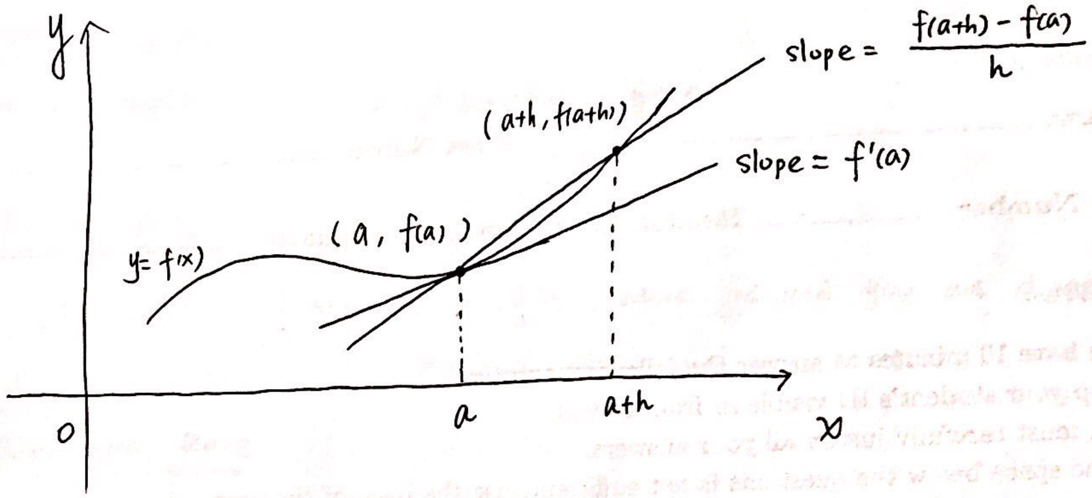
$$f'(a) \quad \text{or} \quad \left.\frac{df}{dx}\right|_{x=a},$$

and it's defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{or} \quad f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a}.$$

If the limit does not exist, we say that f is not differentiable at a. The function is said to be differentiable on an interval if it is differentiable at every point on that interval.

(2) Geometric Interpretation



Thus, we can also see from the graph that

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

2. Calculation Rules for Derivatives.

(i) Power Rule

$$(x^n)' = n \cdot x^{n-1}, \quad n \text{ is constant}, \ n \neq 0.$$

(ii) Product Rule

$$(f(x) \cdot g(x))' = f'(x)\, g(x) + f(x) \cdot g'(x), \text{ when } f(x) \text{ and } g(x) \text{ are differentiable.}$$

(iii) Quotient Rule

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\, g(x) - f(x)\, g'(x)}{[g(x)]^2}, \text{ when } f(x) \text{ and } g(x) \text{ are differentiable; and } g(x) \neq 0.$$

(iv) The Chain Rule

Let $h(x) = f(g(x))$, then

$$h'(x) = f'(g(x)) \cdot g'(x)$$

provided that $f'(g(x))$ and $g'(x)$ are defined.

(v) Implicit Differentiation.

Implicit differentiation is the name of the method of treating y as an implicit function of x, as $y = y(x) = f(x)$, in order to find the derivative of y with respect to x at a given point. The trick is to just differentiate both sides of the equation, and then solve for the derivative we are seeking.

(vi) $\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$.

$\frac{d}{dx}(e^x) = e^x$.

$\frac{d}{dx}(\log x) = \frac{1}{x}$.

(vii) L'Hôpital's Rule

Let $f(x)$ and $g(x)$ be differentiable on an interval containing a, and let $g'(x) \neq 0$ on that interval (except possibly at a). Let $\lim_{x\to a} f(x) = \lim_{x \to a} g(x) = 0$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right-hand side exists or is ∞ or $-\infty$.

This statement of L'Hôpital's Rule also holds if a is replaced by ∞ or $-\infty$, or if, in place of the second sentence, we have $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$ or $-\infty$.

3. Global Extrema.

Definition: $f(x)$ has a global maximum (or an absolute maximum) at a if $f(a) \geq f(x)$ for all x in the domain of $f(x)$, and a global minimum (an absolute minimum) at a if $f(a) \leq f(x)$ for all x in the domain of $f(x)$. The global maximum and the global minimum are called global extrema.

4. Local Extrema

Definition: Let $f(x)$ be a continuous function on a closed interval $[l, r]$. We say that it has a local maximum at a if $f(a) \geq f(x)$ for all x sufficiently close to a; and it has a local minimum at b if $f(b) \leq f(x)$ for all x sufficiently close to b. Local maximum and local minimum are called local extrema.

5. Critical point

We say that $f(x)$ has a critical point at $x=a$ if

$f'(a)=0$ or $f'(a)$ does not exist.

6. Method for Finding the Global Extrema

Suppose we have a continuous function $f(x)$ on a closed interval $[l, r]$. To find the global extremum points we look at:

(a) critical points of $f(x)$, i.e.:

(ai) points where $f'(x)=0$

(aii) points when $f'(x)$ does not exist;

(b) endpoints of $f(x)$, i.e.:

(bi) $f(l)$

(bii) $f(r)$.

Then, we compare the value of them, the smallest ones are global minima, and the largest ones are global maxima.

7. Definition of increasing and decreasing functions:

- We say that $f(x)$ is increasing on an interval if for any pair of points $l<r$ in that interval, $f(l)<f(r)$.

  Also, $f'(x)>0$ on that interval.

- We say that $f(x)$ is decreasing on an interval if for any pair of points $l<r$ in that interval, $f(l)>f(r)$.

  Also, $f'(x)<0$ on that interval.

8. Concavity

Definition: If a differentiable function $f(x)$ has an increasing derivative on an interval, we say it is concave up (convex) on that interval. If it has a decreasing derivative on an interval, we say it is concave down on that interval.

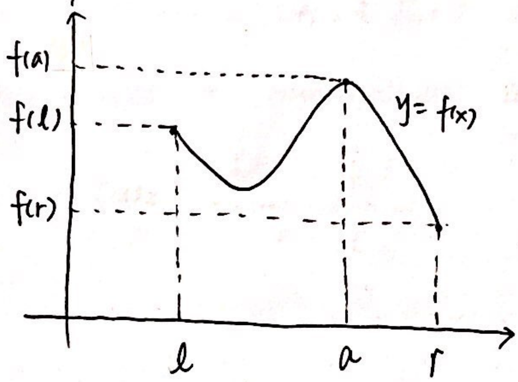
9. Inflection point(s)

Definition: If $f(x)$ changes from concave up to concave down, or vice versa, at a, we say that $f(x)$ has an inflection point at a.

10. Theorems

(i) The Extreme Value Theorem

- Let $f(x)$ be continuous on $[l, r]$. Then $f(x)$ has global extrema on $[l, r]$.
- Geometric interpretation

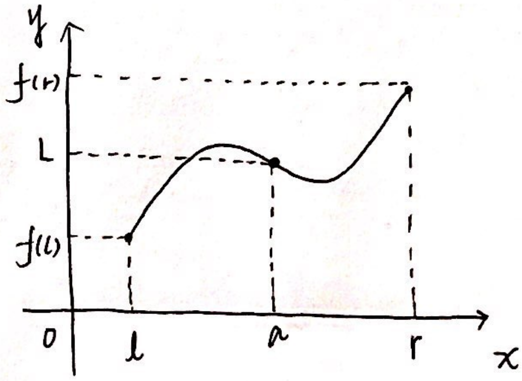


$f(a)$: global maximum

$f(r)$: global minimum.

(ii) The Intermediate Value Theorem

- Let $f(x)$ be continuous on $[l, r]$. Then for any number L between $f(l)$ and $f(r)$, there exists a number a in $[l, r]$ such that $f(a) = L$.
- Geometric Interpretation.



$f(a) = L$, where $f(l) \le L \le f(r)$.

(iii) Interior Extremum Theorem

If $f(x)$ has a local extremum at a; then $f(x)$ has a critical point at a.

(iv). Mean Value Theorem

Let $f(x)$ be continuous on $[l, r]$ and differentiable on (l, r). Then there exists a number a in (l, r) such that

$$f'(a) = \frac{f(r) - f(l)}{r - l}.$$

11. Curve Sketching.

In order to draw a graph of a function, we need to calculate the following:

(1) domain

(2) intercepts

(3) asymptotes

(4) increasing and decreasing intervals and extrema.

(5) concavity and inflection point(s).

Then, based on the information we have, we can sketch the graph.

Questions:

1. Determine whether the derivative of the following function exists at $x=0$.

$$f(x) = \begin{cases} x^3 - 7x^2 & \text{if } x \le 0 \\ x^3 \cos\left(\frac{1}{x}\right) & \text{if } x > 0. \end{cases}$$

You must justify your answer using the definition of a derivative.

2. Let $p(x) = f(x) + g(x)$ for some functions f and g whose derivative exist. Use limit laws and the definition of a derivative to show that $p'(x) = f'(x) + g'(x)$.

3. Differentiate the following functions:

(a) $x(y) = \left(2y + \frac{1}{y}\right) \cdot y^3$.

(b) $T(x) = \sqrt{\dfrac{\sqrt{x}+1}{x^2+3}}$.

(c) $f(x) = (e^x+1)(e^{\sin(3x^2)} + 4x)$

(d) $y^2 + 7x = \log(x+y)$

4. Find an equation of a line that is tangent to both of the curves $y = x^2$ and $y = x^2 - 2x + 2$.

5. Let $f(x) = x\cos(x) - x\sin(x)$. Show that there exists a real number c such that $f'(c) = 0$.

6. A ball is dropped from a height of 49 meters above the ground. The height of the ball at time t is $h(t) = 49 - 4.9t^2$ m. A light, which is also 49 m above the ground, is 10 m to the left of the ball's original position. How fast is the shadow moving one second after the ball is dropped?

7. $f(x) = \frac{2}{3}x^3 - 2x^2 - 30x + 7$. Find all global extrema on the interval $[-4, 0]$.

8. Sketch the graph of the following function:

$$f(x) = x \cdot e^{-x^2},$$

by following the 5 processes.