MATH 100V01 – 2018W
Recitation Notes for Nov. 16th

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1 Finding the Global Extrema

Let $f(x)$ be a continuous function on a closed interval $[l, r]$. Recall that we say $f(x)$ has a ***global maximum*** at a if $f(a) \geq f(x)$ for every $x \in [l, r]$. It has a ***local maximum*** at b if $f(b) \geq f(x)$ for all x sufficiently close to x. ***Global minimum*** and ***local minimum*** are defined similarly. Also, any (local/global) minimum or maximum is called a (local/global) ***extremum***.

Recall also that we have the extreme value theorem.

Theorem 1 (Extreme Value Theorem). *Let $f(x)$ be a continuous function on a closed interval $[l, r]$. Then $f(x)$ has a global extrema on $[l, r]$.*

Note that the extreme value theorem does not provide us with a way to find exactly where the extreme values are. It just ensures that there exists at least one. However, now that we have learned about derivatives, we can find exactly where these extrema are.

Definition 2. We say that $f(x)$ has a ***critical point at*** $x = a$ if $f'(a) = 0$ or $f'(a)$ does not exist.

We have the following theorem to locate the extrema.

Theorem 3 (The Interior Extremum Theorem). *If $f(x)$ has a local extremum at a; then $f(x)$ has a critical point at a.*

Remark. Note that this does not work the other way around. If $f(x)$ has a critical point at a; then, still, $f(x)$ may not have a local extrema at that point. A simple example is the function $f(x) = x^3$ on the closed interval $[-1, 1]$. The point $x = 0$ is a critical point of $f(x)$ since $f'(0) = 0$, but $f(0) = 0$ is neither a local minimum nor a local maximum.

Now, this provides us with a really powerful tool to find the global maximum and minimum of a function.

Method for Finding the Global Extrema. Suppose we have a continuous function $f(x)$ on a closed interval $[l, r]$. To find the global extremum points we look at:

(a) Critical points of $f(x)$, i.e.:

(a.i) Points where $f'(x) = 0$;

(a.ii) Points where $f'(x)$ does not exist;

(b) Endpoints of $f(x)$, i.e.:

(b.i) $f(l)$;

(b.ii) $f(r)$.

Any global extremum of $f(x)$ on $[l, r]$ must be one of these candidates. Hence, we compare them and conclude that the smallest ones are global minima, and the largest ones are global maxima.

2 Problems

Problem 1 (Exercise 2). Give an example of a function $f(x)$ which has the following properties.

(a) $f(x)$ is defined everywhere.

(b) $f(x)$ has a global maximum at $x = 1$, a local minimum (but not a global minimum) at $x = 2$, and no other extrema.

(c) $f(x)$ is discontinuous at $x = 0$, $x = 1$ and $x = 2$, and continuous elsewhere.

Solution. Let us do this systematically. We want a global maximum at $x = 1$, so a simple candidate can be an upside-down parabola, for example the function $x \mapsto -(x-1)^2$. We also want a local minimum at $x = 2$, but discontinuity at $x = 2$ is also required. We can achieve both by moving the point at $x = 2$ slightly downward, by just redefining function, only at $x = 2$, a little below, say -3.

Next, we want a discontinuity at $x = 0$, but this should not create any local extrema. So, redefining the function only at $x = 0$ will not work, because otherwise it will create either a local minima or a maxim at $x = 0$. We cannot also just leave the function undefined there as it has to be defined everywhere. So now, an option can be taking the left piece of the parabola, part where $x < 0$, and moving it a little down, say 1 unit. However, we should be careful about where to put $x = 0$ since we do not want any local extrema at $x = 0$. A little bit of thought will lead to the conclusion that, actually, both options will work.

But also, as a final requirement, we need a discontinuity at $x = 1$, while still keeping at global extremum. We can redifine the function only at 1, by adding, say, 0.5.

Hence,

$$f(x) = \begin{cases} -(x-1)^2 - 1, & \text{if } x < 0; \\ -(x-1)^2, & \text{if } x \geq 0 \text{ and } x \neq 1, 2; \\ 0.5 & \text{if } x = 1; \\ -3, & \text{if } x = 2; \end{cases}$$

is a function that satisfies every requirement above. A graph of the function can be seen in Figure 1.

■

Problem 2 (Exercise 3). Find all global extrema for the following functions in the given domains.

(a) $f(x) = x^3 - 3x^2 - 9x + 3$ on the closed interval $[-2, 1]$.

(b) $f(x) = 2x^3 - x^{2/3}$ on the closed interval $[-1, 1]$.

(c) $f(x) = |\sin(x)|$ on the closed interval $[0, 10]$.

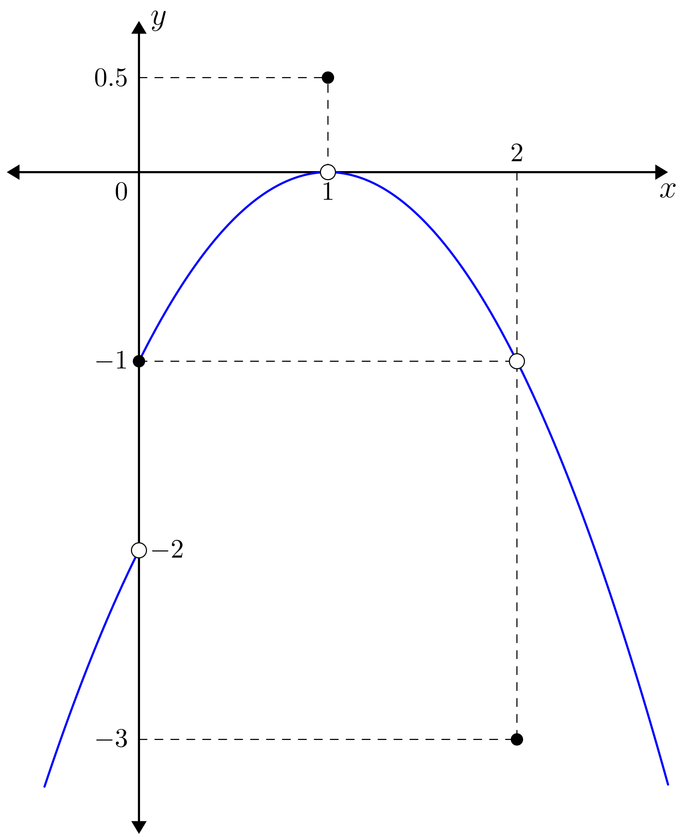

Figure 1: The Plot of the Function $y = f(x)$

Solution. (a) We will use our method. But first, to have an idea, we can plot the function as in Figure 2.

Our function is differentiable in the open interval $(-2, 1)$ since it is a polynomial. So the only critical points in $(-2, 1)$ are where the derivative is zero, i.e.,

$$f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x + 1)(x - 3) = 0$$

Hence, $x = -1$ is the only critical point in $(-2, 1)$.

We also have to look at the endpoints. Our candidates for the global extrema are

- $f(-1) = (-1)^3 - 3 \times (-1)^2 - 9 \times (-1) + 3 = -1 - 3 + 9 + 3 = 8$;
- $f(-2) = (-2)^3 - 3 \times (-2)^2 - 9 \times (-2) + 3 = -8 - 12 + 18 + 3 = 1$; and
- $f(1) = 1 - 3 - 9 + 3 = -8$.

Therefore, the global minimum of $f(x)$ in $[-2, 1]$ is at $x = 1$ with $f(1) = -8$, and the global maximum is at $x = -1$ with $f(-1) = 8$.

(b) The plot the function as in Figure 3.

Again, we use our method. The derivative of our function is

$$f'(x) = 6x^2 - (2/3)x^{-1/3} = 6x^2 - \frac{2}{3\sqrt[3]{x}}.$$

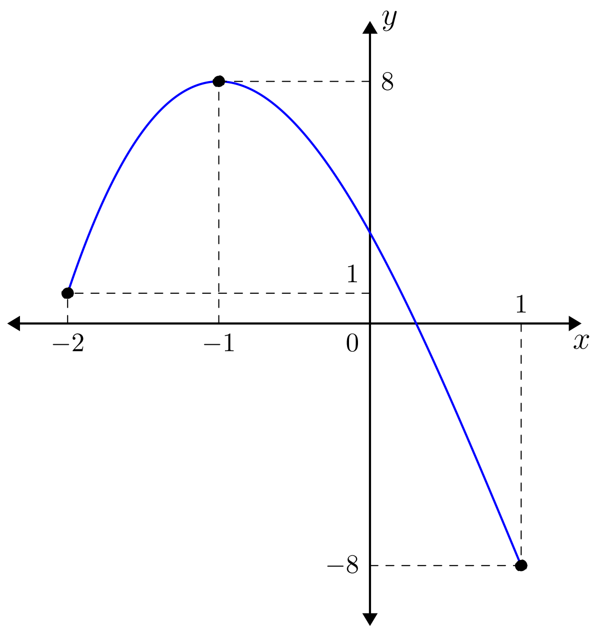

Figure 2: The Graph of the Function $f(x) = x^3 - 3x^2 - 9x + 3$

Observe that $f'(x)$ does not exist at $x = 0$; so $x = 0$ is a critical point. The points where the derivative is zero are where $6x^2 = (2/3)x^{-1/3}$; so, $9x^{7/3} = 1$; and hence, $x = 1/9^{3/7} = 1/3^{6/7}$. Note that the value of this point is approximately 0.39 and so it is in $[-1, 1]$.

Again, our candidates for the global extrema are

- $f(0) = 0$;
- $f\left(\frac{1}{3^{6/7}}\right) = 2\left(\frac{1}{3^{6/7}}\right)^3 - \left(\frac{1}{3^{6/7}}\right)^{2/3} = \frac{2-9}{3^{18/7}} = \frac{-7}{3^{18/7}} \approx -0.42$;
- $f(-1) = -2 - 1 = -3$; and
- $f(1) = 2 - 1 = 1$.

Thus, the global minimum of $f(x)$ in $[-1, 1]$ is at $x = -1$ with $f(-1) = -3$, and the global maximum is at $x = 1$ with $f(1) = 1$.

(c) The graph of this function can be seen in Figure 4. First, let's write this function in a piecewise way:

$$f(x) = \begin{cases} \sin(x), & \text{if } 0 \leq x \leq \pi; \\ -\sin(x), & \text{if } \pi < x \leq 2\pi; \\ \sin(x), & \text{if } 2\pi < x \leq 3\pi; \\ -\sin(x), & \text{if } 3\pi < x \leq 10. \end{cases}$$

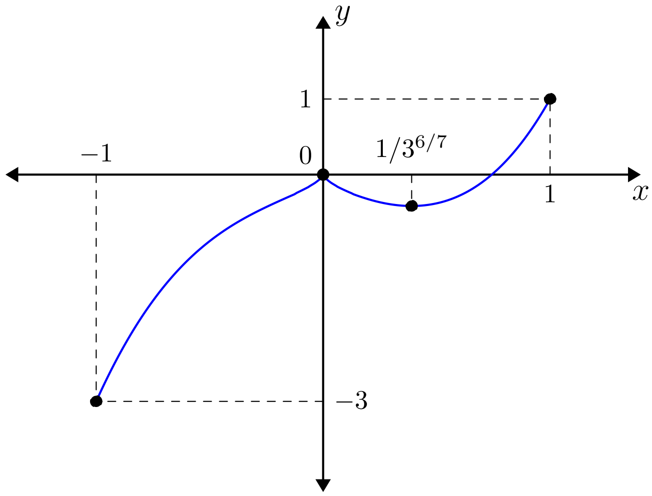

Figure 3: The Graph of the Function $f(x) = 2x^3 - x^{2/3}$

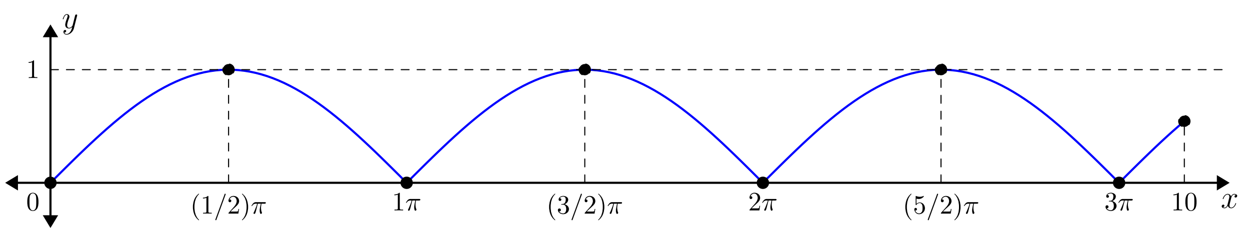

Figure 4: The Graph of the Function $f(x) = |\sin(x)|$

So, after using the definition of the derivative at the endpoints above, we get,

$$f'(x) = \begin{cases} \text{DNE}, & \text{if } x = 0; \\ \cos(x), & \text{if } 0 < x < \pi; \\ \text{DNE}, & \text{if } x = \pi; \\ -\cos(x), & \text{if } \pi < x < 2\pi; \\ \text{DNE}, & \text{if } x = 2\pi; \\ \cos(x), & \text{if } 2\pi < x < 3\pi; \\ \text{DNE}, & \text{if } x = 3\pi; \\ -\cos(x), & \text{if } 3\pi < x < 10; \\ \text{DNE}, & \text{if } x = 10; \end{cases}$$

where, above, "DNE" stands for "does not exist." We see that the derivative is zero at $x = \pi/2, 3\pi/2, 5\pi/2$; and that the derivative does not exist at $x = 0, \pi, 2\pi, 2\pi, 10$.

Thus, our candidates for global extrema—where $f'(x)$ is zero or does not exist, and the endpoints of the interval—are;

$$\begin{cases} \bullet f(0), f(\pi), f(2\pi), f(3\pi) = 0; \\ \bullet f(\pi/2), f(3\pi/2), f(5\pi/2) = 1; \\ \bullet f(10) = |\sin 10|; \end{cases}$$

where $0 < |\sin 10| = -\sin 10 < 1$. So, we can conclude that the function has four global minima at $x = 0, \pi, 2\pi, 3\pi$, with values 0; and three global maxima at $x = \pi/2, 3\pi/2, 5\pi/2$. ■

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