On three-dimensional linear stability of Poiseuille flow of Bingham fluids

Ian Frigaard[a)]
Department of Mathematics and Department of Mechanical Engineering, University of British Columbia, 2324 Main Mall, Vancouver, British Columbia, Canada V6T 1Z4

Cherif Nouar[b)]
LEMTA, CNRS UMR 7563, UHP & INPL, 2, Avenue de la foret de Haye, BP 160, 54504 Vandoeuvre, France

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Plane channel Poiseuille flow of a Bingham fluid is characterized by the Bingham number, B, which describes the ratio of yield and viscous stresses. Unlike purely viscous non-Newtonian fluids, which modify hydrodynamic stability studies only through the dissipation and the basic flow, inclusion of a yield stress additionally results in a modified domain and boundary conditions for the stability problem. We investigate the effects of increasing B on the stability of the flow, using eigenvalue bounds that incorporate these features. As $B\to\infty$ we show that three-dimensional linear stability can be achieved for a Reynolds number bound of form $\mathrm{Re}=O(B^{3/4})$, for all wavelengths. For long wavelengths this can be improved to $\mathrm{Re}=O(B)$, which compares well with computed linear stability results for two-dimensional disturbances [J. Fluid Mech. **263**, 133 (1994)]. It is also possible to find bounds of form $\mathrm{Re}=O(B^{1/2})$, which derive from purely viscous dissipation acting over the reduced domain and are comparable with the nonlinear stability bounds in J. Non-Newt. Fluid Mech. **100**, 127 (2001). We also show that a *Squire-like* result can be derived for the plane channel flow. Namely, if the equivalent eigenvalue bounds for a Newtonian fluid yield a stability criterion, then the same stability criterion is valid for the Bingham fluid flow, but with reduced wavenumbers and Reynolds numbers. An application of these results is to bound the regions of parameter space in which computational methods need to be used.
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I. INTRODUCTION

Shear-thinning yield stress fluids have many industrial and process applications.[1] As with Newtonian fluids, for hydraulic flows it is often necessary to determine the flow regime. Consequently studies of stability have remained a problem of practical interest over the years. Here we consider the three-dimensional linear stability of plane Poiseuille flow of a Bingham fluid. These fluids provide the simplest model of a shear-thinning fluid with a yield stress. That is, if the deviatoric stress somewhere within the fluid does not exceed a certain yield value then the rate of strain is identically zero at that point; unyielded regions behave effectively as rigid bodies, undergoing only linear motion and rotation. The Bingham model is both widely studied and widely used in various industrial applications.

In the limit of zero yield stress, one recovers a Newtonian fluid. For Newtonian fluids, linear stability analyses of Poiseuille flows are quite classical. Historically, much of the interest in studying these flows has come from the large discrepancies that exist between the critical Reynolds numbers computed for linear instability, for fully nonlinear stability and those observed experimentally. For a plane channel flow, Squire's theorem[2] implies that three-dimensional infinitesimal disturbances are more stable than two-dimensional disturbances for all wavenumbers. Accurate numerical solution of the corresponding Orr–Sommerfeld problem gives the linear stability limit of $\mathrm{Re}^*=5772.24$, see Ref. 3. Here Re^* is based on the maximum axial velocity and half width of the channel. The fully nonlinear energy stability limit is $\mathrm{Re}^*=99.207$, see Refs. 4 and 5. Experimental observations of instability occur at $\mathrm{Re}^*\approx1000$. Early attempts at consolidating the discrepancy between different critical Reynolds numbers focused on weakly nonlinear stability theory, e.g., Refs. 6 and 7. More recently, it has been again linear analyses, in various forms, that have been successful in explaining transitional phenomena, e.g., Refs. 8–12. This partly explains the persistent interest in linear stability for understanding a subcritical phenomenon.

Bingham fluids fall also into the broad category of generalized Newtonian fluids. Such fluids are inelastic but have an effective viscosity $\eta(\boldsymbol{u})$ that varies with the second invariant of the rate of strain tensor, $\dot{\gamma}(\boldsymbol{u})$, i.e., the deviatoric stress tensor is

$$\tau_{ij}(\boldsymbol{u})=\eta(\boldsymbol{u})\,\dot{\gamma}_{ij}(\boldsymbol{u}).$$

Here $\boldsymbol{u}$ denotes an arbitrary velocity field. If we consider linear stability of a simple shear flow $\boldsymbol{U}=[U(y),0,0]$, the momentum equations for the disturbance, denoted $(\boldsymbol{u},p)$ are

a)Electronic address: frigaard@math.ubc.ca
b)Visiting at Department of Mathematics and Department of Mechanical Engineering, University of British Columbia. Electronic address: cherif.nouar@ensem.inpl-nancy.fr

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$$u_t+Uu_x+vU_y=-p_x+\frac{1}{\mathrm{Re}}\left[\Delta u+\frac{\partial}{\partial x_j}[(\eta(\boldsymbol{U})-1)\dot{\gamma}_{xj}(\boldsymbol{u})]+\frac{\partial}{\partial y}\left[\dot{\gamma}(\boldsymbol{U})\frac{\mathrm{d}\eta}{\mathrm{d}\dot{\gamma}}(\boldsymbol{U})\dot{\gamma}_{xy}(\boldsymbol{u})\right]\right], \quad (1)$$

$$v_t+Uv_x=-p_y+\frac{1}{\mathrm{Re}}\left[\Delta v+\frac{\partial}{\partial x_j}[(\eta(\boldsymbol{U})-1)\dot{\gamma}_{yj}(\boldsymbol{u})]+\frac{\partial}{\partial x}\left[\dot{\gamma}(\boldsymbol{U})\frac{\mathrm{d}\eta}{\mathrm{d}\dot{\gamma}}(\boldsymbol{U})\dot{\gamma}_{yx}(\boldsymbol{u})\right]\right], \quad (2)$$

$$w_t+Uw_x=-p_z+\frac{1}{\mathrm{Re}}\left[\Delta w+\frac{\partial}{\partial x_j}[(\eta(\boldsymbol{U})-1)\dot{\gamma}_{zj}(\boldsymbol{u})]\right], \quad (3)$$

for suitably defined dimensionless variables, where Re is the Reynolds number. For a given flow domain and boundary conditions (1)–(3) are used to define an eigenvalue problem for the phase velocity C of a normal mode of the disturbance, each component of which has form

$$\psi(x,y,z,t)=\varphi(y)\mathrm{e}^{i(\alpha x+\beta z-Ct)},$$

where α and β are wavenumbers. Thus, $C=C(\mathrm{Re},\alpha,\beta)$, and if the imaginary part of the phase velocity $\mathrm{Im}(C)$ is positive, the disturbance grows exponentially in time.

If the fluid is a purely viscous generalized Newtonian fluid (i.e., meaning η is a continuous positive bounded function of $\dot{\gamma}\in[0,\infty)$), the departure from Newtonian fluid behavior is captured physically in two ways only.

(1) The basic shear velocity $U(y)$ can be different from the Newtonian velocity, and thus the inertial terms on the left hand side of (1)–(3) will be changed. The term vU_y represents transfer of energy from the basic flow to the disturbance and is of primary importance, whereas the other terms are purely advective and play a lesser role.
(2) The change in viscous dissipation is represented by the terms on the right hand side of (1)–(3) that are additional to the Newtonian terms, Δu_i. These terms are generally nonlinear in $\dot{\gamma}(\boldsymbol{U})$. The terms multiplied by $[\eta(\boldsymbol{U})-1]$ lead physically to a dissipation rate which is isotropic with respect to $\boldsymbol{u}$, but we note that there are also additional terms in the x- and y-momentum equations. Thus, the effects of nonconstant η on the viscous dissipation are in general anisotropic (i.e., we do not simply have $[\eta(\boldsymbol{U})-1]\Delta u_i$ as the *non-Newtonian* dissipation).

Apart from the subtlety of the anisotropy, one could say that the above features are self-evident. One might conjecture that the stability characteristics of different fluids for which the respective functions η are *close*, would be similar. It is almost certainly possible to prove such continuity results mathematically, for restricted classes of η, since these represent perturbations of the spectra of linear operators.

If we now consider a generalized fluid with a yield stress (and for simplicity, the plane Poiseuille flow Bingham fluid), the above two features evidently hold true. The basic shear velocity is characterized dimensionlessly by the Bingham number, B, which is the ratio of yield and viscous stresses. The nonconstant viscous terms also depend parametrically only on B. The dependence on B is smooth and the physical way in which a Bingham fluid modifies the linear stability is in these ways completely analogous to that of other purely viscous fluids. However, the stability problem for a yield stress fluid also exhibits two striking differences to that for a purely viscous fluid.

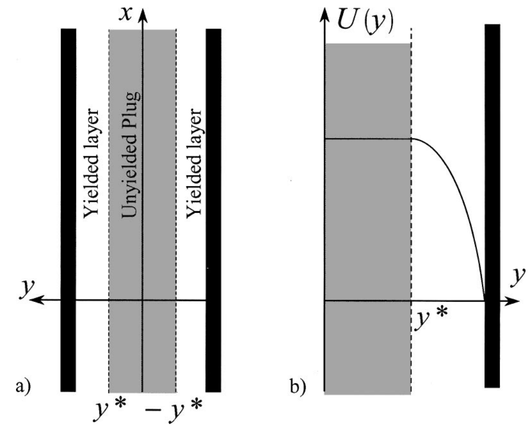

FIG. 1. Schematic of the plane Poiseuille flow of a Bingham fluid: (a) Unyielded and unyielded layers; (b) $U(y)$.

(1) The basic shear velocity $U(y)$ is yielded only in two layers close to the walls of the channel, bounded by two yield surfaces at $y=\pm y^*$, see Fig. 1. When considering a linear perturbation, the yield surfaces are also linearly perturbed. In common with other problems where interfacial perturbations occur, the interface perturbation is linearized onto the basic flow domain, $|y|\in(y^*,1)$, for the linear stability problem. Thus, (1)–(3) and the resulting eigenvalue problems for stability are solved on a modified domain: $|y|\in(y^*,1)$, although the basic flow is defined on $[-1,1]$. The position of the yield surfaces is a function only of B.
(2) In performing the linearization leading to (1)–(3), it is necessary also to: (i) Linearize the equations that define the yield surface; (ii) linearize the integral momentum equations for the motion of the unyielded *plug* region. A combination of these linearizations leads to boundary conditions for the perturbation that are to be satisfied at $y=\pm y^*$. Since for a yield stress fluid $\eta\to\infty$ at a yield surface, some care is required in deriving the boundary conditions at $y=\pm y^*$, as explained for example in Ref. 13.

Thus, the effect of a yield stress on the flow stability is complex, both physically and mathematically, even at an intuitive level. Although it is true that as $B\to 0$ the Newtonian problem is recovered, this is of little practical interest for fluids with a significant yield stress. For such fluids (i.e., finite $B>0$), the stability eigenvalue problem is posed on a different domain and with different boundary conditions to its Newtonian counterpart. This is not true of purely viscous non-Newtonian fluids.

The above explains some of the physical interest in studying this type of stability problem. Although mathematically there is a single parameter B and the only aim is to define the influence of B on the critical Reynolds number, this influence arises from four different effects: (i) Variation of the inertial terms; (ii) variation of the dissipative terms; (iii) variation of the domain over which the competition between energy transfer and dissipation can take place; (iv) variation of the boundary conditions for the perturbation. It is evidently of interest to understand how these individual effects contribute.

Stability of plane Poiseuille flow of a Bingham fluid to two-dimensional linear perturbations has been studied in Ref. 13, via solution of the Orr–Sommerfeld problem. As $B\rightarrow\infty$, the critical Reynolds number for linear stability appears to approach a linear increase with the Bingham number. For Bingham fluids (and other nonlinearly viscous fluids), there is no general result equivalent to Squire's theorem (although results of this nature can be established by making somewhat unphysical restrictions on the type of perturbation, see, e.g., Ref. 14). Thus, three-dimensional stability remains a relevant and unsolved problem for the plane channel flow. Recently, the authors have approached the problem of nonlinear stability of Poiseuille flow, using energy methods. Nonlinear stability bounds have been derived for both plane channel and Hagen–Poiseuille flows. In these bounds the critical Reynolds number behaves like $\mathrm{Re}=O(B^{1/2})$ as $B\rightarrow\infty$, see Ref. 15.

This paper deals with the three-dimensional linear stability problem. The aim is to derive bounds on the disturbance growth rate $\mathrm{Im}(C)$ that depend on the parameters of the eigenvalue problem, $(\mathrm{Re},B,\alpha,\beta)$. For Newtonian fluids, $(B=0)$, an upper bound for the growth rate $\mathrm{Im}(C)$ for a two-dimensional linear disturbance was given for the first time by Synge.[16] This result has been improved in Refs. 17 and 18 and the method extended to other flows, e.g., the parallel boundary layer flow in a round pipe is derived in Ref. 19. In the present paper we follow essentially Synge's method, with the additional complication of the Bingham fluid, and we also consider three-dimensional disturbances. The purpose of these bounds is threefold. First, these bounds are extremely valuable when undertaking computation of the three-dimensional linear stability problem to determine actual values for marginal instability, since they allow one to eliminate large regions of $(\mathrm{Re},B,\alpha,\beta)$-parameter space as being stable. Second, although such bounds are conservative when compared with numerical solution, they do give an analytical expression for the parametric dependency of the marginal stability curves. Third, since we may include in our method each of the four ways in which the Bingham fluid modifies the Newtonian problem, there is some hope that the parametric dependency we derive will be sharp, even when the bound itself is not.

Linear stability studies of purely viscous generalized Newtonian fluids are surprisingly few. Gupta[20] has investigated the two-dimensional linear stability of power fluids as a particular case of a shear-thinning electrorheological fluid model. Numerical results are given for behavior flow index $n=0.7$, 0.9, 1, 1.1, 1.3, and indicate that the critical Reynolds number increases with shear thickening of the fluid, which is perhaps intuitive. We note that the non-Newtonian terms in (1)–(3) will be singular at the channel center for a shear-thinning power-law fluid $n<1$. It is unclear how this has been addressed in Ref. 20. Other than Ref. 20, we have been unable to find any treatment of linear instability of a single power-law fluid. For multilayer flows the situation is quite different. For example, Refs. 21 and 22 consider two-layer Couette flow, Refs. 23 and 24 consider two-layer Poiseuille flow and Ref. 25 considers two-layer boundary-layer flow. Additionally, Ref. 26 considers two-layer Poiseuille flows of Carreau–Yasuda fluids. Each of these stability studies is of course quite different from that of a single fluid shear flow, since the interface can become linearly unstable at relatively low Reynolds numbers.

In contrast to the above class of non-Newtonian fluids, the stability of Poiseuille flow of viscoelastic fluids has been considered by several authors. We do not intend to fully review this work here, for a number of reasons. First, viscoelasticity is quite different from visco-plasticity. We have explained above what are the unique physical features of a visco-plastic hydrodynamic stability problem, and intend to keep this focus. Second, reported results for visco-elastic fluids come from a wide range of different constitutive models, which is rather confusing to interpret. Indeed, where one finds generality in different visco-elastic constitutive models is for slow, nearly steady low shear rate flows, e.g., every (simple) visco-elastic fluid model becomes a second-order fluid asymptotically in these limits. Obviously, such flows differ from high-speed, high shear rate flows that are characteristic of Newtonian transition. As far as we may make general statements, it appears that visco-elastic flows are stable at low Reynolds numbers, but that elasticity can be strongly destabilizing at high Reynolds numbers, see e.g., Refs. 27–31. Finally, we remark that a whole range of different instabilities (of a structural nature) may occur with visco-elastic fluid models, e.g., extrusion instabilities. Thus, the reader is best advised to look elsewhere for a comprehensive review of this area.

II. THE EIGENVALUE PROBLEM FOR PLANE POISEUILLE FLOW

The constitutive model that we consider throughout the paper is a Bingham fluid. These fluids are characterized by a density $\hat{\rho}$, a yield stress $\hat{\tau}_0$ and a plastic viscosity $\hat{\mu}_0$. The geometry of the plane Poiseuille flow is a three-dimensional plane channel formed by two infinite flat plates at $\hat{y}=\pm L$. It is assumed that there is an imposed dimensional pressure gradient in the $\hat{x}$-direction, say $\hat{p}=-\hat{P}_0\hat{x}$, with an appropriate choice of coordinate origin. We nondimensionalize the Navier–Stokes equations using a length-scale $\hat{L}$, a velocity scale $\hat{U}_0$, a time scale $\hat{L}/\hat{U}_0$, and a pressure–stress scale $\hat{\rho}\hat{U}_0^2$. The velocity scale $\hat{U}_0$ is chosen to be the mean speed of the basic flow in the $\hat{x}$-direction, averaged across the channel.

Using this nondimensionalization, and omitting the hat notation for dimensionless variables, scaled constitutive laws for the fluid are

$$\boldsymbol{\tau}=\frac{1}{\text{Re}}\eta(\dot{\gamma})\dot{\boldsymbol{\gamma}}\Leftrightarrow\tau\geqslant\frac{B}{\text{Re}},\tag{4}$$

$$\dot{\gamma}=0\Leftrightarrow\tau<\frac{B}{\text{Re}},\tag{5}$$

$$\eta(\dot{\gamma})=1+\frac{B}{\dot{\gamma}},\tag{6}$$

where $\dot{\boldsymbol{\gamma}}$ and $\boldsymbol{\tau}$ are the rate-of-strain and deviatoric stress tensors, respectively, with second invariants $\dot{\gamma}$ and τ, respectively. These are defined by

$$\dot{\gamma}=[\tfrac{1}{2}\dot{\gamma}_{ij}\dot{\gamma}_{ij}]^{1/2},\qquad \tau=[\tfrac{1}{2}\tau_{ij}\tau_{ij}]^{1/2},\tag{7}$$

where $\dot{\gamma}_{ij}=u_{i,j}+u_{j,i}$. These flows are characterized by two dimensionless groups, the Reynolds number, Re, and Bingham number B:

$$\text{Re}=\frac{\hat{\rho}\hat{U}_0\hat{L}}{\hat{\mu}_0},\qquad B=\frac{\hat{\tau}_0\hat{L}}{\hat{\mu}_0\hat{U}_0}.\tag{8}$$

In the configuration described there exists the following steady Poiseuille flow solution, $(P,\boldsymbol{U})=(P(x),U(y),0,0)$, of the full Navier–Stokes equations:

$$P(x)=-\frac{B}{\text{Re}\,y^*}x,\tag{9}$$

$$U(y)=\begin{cases}\dfrac{B(1-y^*)^2}{2y^*}, & 0\leqslant|y|<y^*,\\[2ex] \dfrac{B(1-y^*)^2}{2y^*}\left[1-\left(\dfrac{|y|-y^*}{1-y^*}\right)^2\right], & y^*\leqslant|y|\leqslant1,\end{cases}\tag{10}$$

where $y^*=\hat{\tau}_0/\hat{\tau}_w$ is the position of the yield surface and $\hat{\tau}_w$ denotes the wall shear stress. Thus, the basic Poiseuille flow (10) consists of an unyielded region $|y|<y^*$ in the channel center, see Fig. 1, where the viscosity is effectively infinite, bounded by two yielded regions for $y^*\leqslant|y|\leqslant1$, in which there is a nonlinear variation in the effective viscosity. The position of the yield surfaces, $y^*<1$, is found as the solution of

$$(y^*)^3-3y^*\left[1+\frac{2}{B}\right]+2=0,\tag{11}$$

which depends only on the Bingham number, B. This is straightforward to find numerically, see, e.g., Ref. 15. Later we shall be interested in the stability behavior of the flow for large B. In the limit $B\to\infty$, the following expression may be derived:

$$y^*(B)\sim1-\frac{\sqrt{2}}{B^{1/2}}+\frac{2}{3B}+O(B^{-3/2}).\tag{12}$$

Normal mode equations

We consider an infinitesimal disturbance $(\boldsymbol{u},p)$ to the primary flow $(\boldsymbol{U},P)$ described by (9) and (10) above. The disturbance is assumed periodic in both x and z-directions. The linearized momentum equations for $(\boldsymbol{u},p)$ are (1)–(3), which are supplemented with

$$\nabla\cdot\boldsymbol{u}=0.\tag{13}$$

To derive the eigenvalue problem, we assume that the solution can be represented in terms of normal modes, of form

$$(u,v,w,p,h)=(u(y),v(y),w(y),p(y),h)e^{i(\alpha x+\beta z-Ct)},$$

where α and β are the wavenumbers and $C=C_r+iC_i$ is the complex wave speed. The linearized disturbance equations (1)–(3), (13), and boundary conditions have been derived in Ref. 13. In the case that $\alpha=\beta=0$, the perturbation is one-dimensional and the mode can be shown to be linearly stable, see Ref. 13. Thus, we ignore this special case. Denoting $D=\mathrm{d}/\mathrm{d}y$, the linearized equations for the normal modes are found by substituting into (1)–(3), (13):

$$0=i[\alpha u+\beta w]+Dv,\tag{14}$$

$$-iCu=-i\alpha Uu-vDU-i\alpha p+\frac{1}{\text{Re}}[D^2-(\alpha^2+\beta^2)]u+\frac{B}{\text{Re}}\left[\frac{-(\alpha^2+\beta^2)u-i\alpha Dv}{|DU|}\right],\tag{15}$$

$$-iCv=-i\alpha Uv-Dp+\frac{1}{\text{Re}}[D^2-(\alpha^2+\beta^2)]v+\frac{B}{\text{Re}}\left[D\left(\frac{2Dv}{|DU|}\right)+\frac{-\beta^2v-i\beta Dw}{|DU|}\right],\tag{16}$$

$$-iCw=-i\alpha Uw-i\beta p+\frac{1}{\text{Re}}[D^2-(\alpha^2+\beta^2)]w+\frac{B}{\text{Re}}\left[D\left(\frac{i\beta v+Dw}{|DU|}\right)-\frac{i\beta Dv+(\alpha^2+\beta^2)w}{|DU|}\right].\tag{17}$$

The boundary conditions at the wall are

$$u(1)=v(1)=w(1)=0,\tag{18}$$

and at the yield surface:

$$u(y^*)=0,\qquad Du(y^*)=-hD^2U(y^*),\tag{19}$$

$$v(y^*)=0,\qquad Dv(y^*)=0,\tag{20}$$

$$w(y^*)=0,\qquad Dw(y^*)=0.\tag{21}$$

While (18) is fairly obvious, (19)–(21) are not so. The Dirichlet conditions in (19)–(21) come from consideration of the linear momentum of the unyielded plug region. The Neumann conditions come from linearization of the condition $\dot{\gamma}_{ij}(\boldsymbol{U}+\boldsymbol{u})=0$, at the perturbed yield surface, onto the unperturbed yield surface position. It may at first appear to the reader that the problem is over-specified (i.e., we have nine boundary conditions for a sixth-order problem). However, this is not the case. The condition on Du in fact defines the amplitude of the yield surface perturbation h, i.e., it is a condition for h, not for $\boldsymbol{u}$. The conditions on Dv and Dw are necessary due to the singular behavior of $|DU|^{-1}$ in the non-Newtonian part of (15)–(17), i.e., these conditions ensure that (15)–(17) are well-defined as $y\to y^*$. As is well known, for eigenvalue problems with singular coefficients it is possible to find eigenfunctions that are both regular and singular at the singular points (e.g., Bessel's equation). The conditions on Dv and Dw simply specify that it is the regular eigenfunctions that are of interest.

Note that problems (14)–(21) is only defined on the yielded portion of the half-channel $y \in [y^*,1]$. Physically, an infinitesimal linear perturbation does not succeed in perturbing an infinitely long unyielded region of fluid with finite width. Hence, the linear stability problems in the two yielded regions in fact decouple completely, i.e., we may define an equivalent (and independent) problem on $[-y^*,-1]$. A complete derivation of (14)–(21) is given in Ref. 13, to which the reader is referred for further details.

III. AN UPPER BOUND ON THE GROWTH RATE

To obtain our growth rate bound, we proceed essentially as in Ref. 16. We multiply the normal mode equations for u, v and w by their respective complex conjugates u^*, v^*, w^*, and integrate between y^* and 1. The resulting equations are summed. On using the continuity equation and after some minor manipulations, we obtain the identities:

$$C_r\langle|\mathbf{u}|^2\rangle = \alpha\langle U|\mathbf{u}|^2\rangle + \langle(v_i u_r - v_r u_i)DU\rangle \tag{22}$$

and

$$C_i\langle|\mathbf{u}|^2\rangle = \mathcal{I}(\mathbf{u}) - \mathcal{V}(\mathbf{u}) - \mathcal{B}(\mathbf{u}), \tag{23}$$

where $\mathcal{I}(\mathbf{u})$, $\mathcal{V}(\mathbf{u})$, and $\mathcal{B}(\mathbf{u})$ denote the inertial, viscous and *Bingham* terms, defined by

$$\mathcal{I}(\mathbf{u}) = -\langle(u_r v_r + u_i v_i)DU\rangle, \tag{24}$$

$$\mathcal{V}(\mathbf{u}) = \frac{1}{\mathrm{Re}}\langle|D\mathbf{u}|^2 + (\alpha^2+\beta^2)|\mathbf{u}|^2\rangle, \tag{25}$$

$$\mathcal{B}(\mathbf{u}) = \frac{B}{\mathrm{Re}}\left\langle\frac{3|Dv|^2 + (\alpha^2+\beta^2)(|u|^2+|w|^2)}{|DU|}\right\rangle + \frac{B}{\mathrm{Re}}\left\langle\frac{|Dw + i\beta v|^2}{|DU|}\right\rangle, \tag{26}$$

with notation $|u|^2 = u_r^2 + u_i^2$, $|\mathbf{u}|^2 = |\mathrm{u}|^2 + |\mathrm{v}|^2 + |\mathrm{w}|^2$, and $\langle\cdot\rangle = \int_{y^*}^1(\cdot)\mathrm{d}y$.

Our aim now is to bound C_i, and our primary concern is to understand how such bounds vary with B, rather than to obtain sharp bounds. The reason for this bias is mainly due to the fact that bounds obtained from expressions such as (23), tend to anyway give conservative predictions of the marginal stability curves (this is at least the case for Newtonian fluids). Although not sharp, when considered in comparison to directly computed values, there is however some hope that the parametric dependence of the bounds may be sharp, i.e., because the inaccuracy comes from the use of functional inequalities, not from neglecting terms representing different physical (parametric) effects.

More specifically, in the introduction we have explained how the influence of B arises from four different effects: (i) Variation of the inertial terms; (ii) variation of the dissipative terms; (iii) variation of the domain over which the competition between energy transfer and dissipation can take place; (iv) variation of the boundary conditions for the perturbation. In solving the actual eigenvalue problem, as in Ref. 13, all four effects are present explicitly. In deriving (23) we incorporate directly (i)–(iii) in the bound. For (iv), the effect of having different boundary conditions on an eigenvalue problem is harder to gauge. Here we will use rather general functional inequalities in our analysis (e.g., triangle inequality, Cauchy–Schwarz inequality, Poincaré inequality, etc.). As previously explained, the *additional* boundary conditions in our problem are simply to ensure regularity of the eigenfunctions and do not over-specify the problem. Thus, although the additional boundary conditions may modify the constants in these inequalities, the inequalities themselves are unchanged. Thus, we believe that our analysis contains all the physical features of the problem and should be able to produce parametrically sharp bounds. This is our goal.

A. Preliminary bounds

The overall aim here is to reduce each of the terms in (23) to an expression involving only integrals of $\boldsymbol{u}^2$. To do this we employ the Cauchy–Schwarz, Poincaré and triangle inequalities, all of which may be found in standard references, e.g., Ref. 32. We bound the inertial term in (23) as follows:

$$\begin{aligned}
\mathcal{I}(\mathbf{u}) &= -\frac{B}{y^*}\int_{y^*}^1 (y-y^*)u_r(y)\int_{y^*}^y[\alpha u_i(\tilde{y}) + \beta w_i(\tilde{y})]\mathrm{d}\tilde{y}\mathrm{d}y - \frac{B}{y^*}\int_{y^*}^1 (y-y^*)u_i(y)\int_{y^*}^y[\alpha u_r(\tilde{y}) + \beta w_r(\tilde{y})]\mathrm{d}\tilde{y}\mathrm{d}y \\
&\le \frac{B(1-y^*)^3}{y^*}\left[2\alpha\int_0^1|u_i(\xi)|\mathrm{d}\xi\int_0^1|u_r(\xi)|\mathrm{d}\xi + \beta\int_0^1|u_i(\xi)|\mathrm{d}\xi\int_0^1|w_r(\xi)|\mathrm{d}\xi + \beta\int_0^1|u_r(\xi)|\mathrm{d}\xi\int_0^1|w_i(\xi)|\mathrm{d}\xi\right] \\
&\le \frac{B(1-y^*)^3}{y^*}\left[2\alpha\left(\int_0^1|u_i|^2\mathrm{d}\xi\int_0^1|u_r|^2\mathrm{d}\xi\right)^{1/2} + \beta\left(\int_0^1|u_i|^2\mathrm{d}\xi\int_0^1|w_r|^2\mathrm{d}\xi\right)^{1/2} + \beta\left(\int_0^1|u_r|^2\mathrm{d}\xi\int_0^1|w_i|^2\mathrm{d}\xi\right)^{1/2}\right] \\
&\le \frac{B(1-y^*)^3}{2y^*}\left[2\alpha\int_0^1|u|^2\mathrm{d}\xi + \beta\int_0^1|u|^2 + |w|^2\mathrm{d}\xi\right].
\end{aligned} \tag{27}$$

In the above on successive lines, we have substituted for DU, used the continuity equation, mapped to $\xi=(y-y^*)/(1-y^*)$, used the Cauchy–Schwarz inequality, used the triangle inequality. Using the same mapping for $y\mapsto\xi$ and the Poincaré inequality, the viscous term $\mathcal{V}(\mathbf{u})$ is bounded as follows:

$$\mathcal{V}(\mathbf{u})=\frac{(1-y^*)}{\mathrm{Re}}\int_0^1(\alpha^2+\beta^2)|\mathbf{u}|^2+\frac{1}{(1-y^*)^2}\left|\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\xi}\right|^2\mathrm{d}\xi, \\ \geqslant\frac{\pi^2+(1-y^*)^2(\alpha^2+\beta^2)}{\mathrm{Re}(1-y^*)}\int_0^1|u|^2+|w|^2\mathrm{d}\xi. \quad (28)$$

For the yield stress term $\mathcal{B}(\mathbf{u})$, we first insert an upper bound for $|DU|$:

$$\mathcal{B}(\mathbf{u})\geqslant\frac{y^*}{(1-y^*)\mathrm{Re}}\int_0^{y^*}3|Dv|^2+(\alpha^2+\beta^2)(|u|^2+|w|^2) \\ +|Dw+i\beta v|^2\mathrm{d}y. \quad (29)$$

Now transform $|Dv|^2$ using the continuity equation:

$$3|Dv|^2=3[\alpha^2|u|^2+\beta^2|w|^2+\alpha\beta(uw^*+u^*w)],$$

multiply out the last term, integrate by parts to transfer the derivative to v, then use the continuity equation

$$\int_0^{y^*}|Dw+i\beta v|^2\mathrm{d}y=\int_0^{y^*}|Dw|^2+\beta^2|v|^2 \\ -\alpha\beta(uw^*+u^*w)-2\beta^2|w|^2\mathrm{d}y.$$

Finally, apply the mapping $y\mapsto\xi$ and insert the above:

$$\mathcal{B}(\mathbf{u})\geqslant\frac{y^*}{\mathrm{Re}}\int_0^1 3[\alpha^2|u|^2+\beta^2|w|^2+\alpha\beta(uw^*+u^*w)]+(\alpha^2+\beta^2)(|u|^2+|w|^2) \\ +(1-y^*)^{-2}\left|\frac{\mathrm{d}w}{\mathrm{d}\xi}\right|^2+\beta^2|v|^2-\alpha\beta(uw^*+u^*w)-2\beta^2|w|^2\mathrm{d}\xi \\ \geqslant\frac{y^*}{\mathrm{Re}}\int_0^1(4\alpha^2+\beta^2)|u|^2+(\alpha^2+2\beta^2)|w|^2+4\alpha\beta(u_rw_r+u_iw_i)\mathrm{d}\xi \\ \geqslant\frac{y^*}{\mathrm{Re}}\int_0^1(2[2-\lambda^2]\alpha^2+\beta^2)|u|^2+\left(\alpha^2+2\left[1-\frac{1}{\lambda^2}\right]\beta^2\right)|w|^2\mathrm{d}\xi, \quad (30)$$

for any $\lambda\in(1,2)$. The latter is simply completing the square.

B. Uniform bound for all wavenumbers

We now seek a bound on C_i that is independent of the wavelength of the perturbation. To this end, we define

$$\alpha=\delta\cos\phi,\quad \beta=\delta\sin\phi:\quad \phi\in[0,\pi/2]. \quad (31)$$

Inequalities (27), (28), and (30) become

$$\mathcal{I}(\mathbf{u})\leqslant\frac{B(1-y^*)^3k_1\delta}{2y^*}\int_0^1|u|^2+|w|^2\mathrm{d}\xi, \quad (32)$$

$$\mathcal{V}(\mathbf{u})\geqslant\frac{\pi^2+(1-y^*)^2\delta^2}{\mathrm{Re}(1-y^*)}\int_0^1|u|^2+|w|^2\mathrm{d}\xi, \quad (33)$$

$$\mathcal{B}(\mathbf{u})\geqslant\frac{2y^*\delta^2k_2}{\mathrm{Re}}\int_0^1|u|^2+|w|^2\mathrm{d}\xi, \quad (34)$$

where

$$k_1=2\cos(\tan^{-1}(\tfrac{1}{2}))+\sin(\tan^{-1}(\tfrac{1}{2}))=2.360\,68..., \quad (35)$$

$$k_2=\frac{3-\sqrt{5}}{2}. \quad (36)$$

The value of k_2 is obtained by choosing $\lambda^2=(1+\sqrt{5})/2$. Combining all the above we have

$$C_i\leqslant F(\delta)\frac{\int_0^1|u|^2+|w|^2\mathrm{d}\xi}{\int_0^1|\mathbf{u}|^2\mathrm{d}\xi}, \quad (37)$$

$$F(\delta)=\frac{B(1-y^*)^2k_1\delta}{2y^*}-\frac{\pi^2}{\mathrm{Re}(1-y^*)^2} \\ -\frac{\delta^2}{\mathrm{Re}}\left[1+\frac{2y^*k_2}{1-y^*}\right]. \quad (38)$$

The (marginal) linear stability bounds for which $F(\delta)=0$ are shown in Fig. 2, for different values of B. It is interesting to

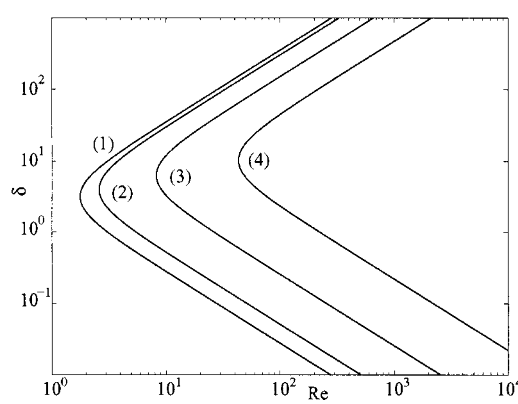

FIG. 2. Linear stability bounds for the plane Poiseuille flow of Bingham fluid: (1) $B=0$; (2) $B=1$; (3) $B=10$; (4) $B=100$.

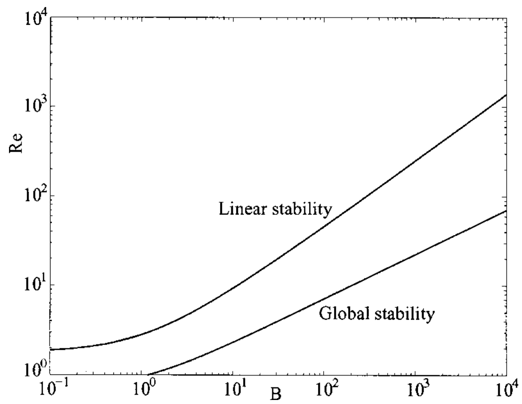

FIG. 3. Critical Reynolds number $\mathrm{Re}_{\mathrm{lin1}}$ as a function of B.

observe that there is not a cut-off value of δ, above which there is always linear stability. This is contrary to the case with actual computed marginal stability curves, for two-dimensional disturbances, at each fixed value of B, see Ref. 13.

The function $F(\delta)$ is maximized when $\delta=\delta_{\max}$:

$$\delta_{\max}=\frac{k_1\,\mathrm{Re}\,B(1-y^*)^2}{4y^*\left[1+\dfrac{2y^*k_2}{1-y^*}\right]}. \tag{39}$$

Thus, choosing Re:$F(\delta)\leqslant F(\delta_{\max})<0$, we have linear stability for all wavelengths if

$$\mathrm{Re}<\frac{4\pi y^*\left[1+\dfrac{2y^*k_2}{1-y^*}\right]^{1/2}}{k_1B(1-y^*)^3}=\mathrm{Re}_{\mathrm{lin1}}. \tag{40}$$

Figure 3 shows the evolution of $\mathrm{Re}_{\mathrm{lin1}}$ as a function of B. We have also shown the result obtained from the nonlinear stability analysis in Ref. 15.

Behavior at large B

For computational purposes (i.e., principally in order to limit the parameter space in which one has to search for unstable solutions), we are more interested in the behavior of $\mathrm{Re}_{\mathrm{lin1}}$ for large B than at small B. Using (12) we observe that

$$\mathrm{Re}_{\mathrm{lin1}}\sim\frac{2^{3/4}\pi k_2^{1/2}}{k_1}B^{3/4}\quad\text{as}\quad B\to\infty, \tag{41}$$

which is independent of wavenumber. For each fixed B, we can also use (38) directly to give sufficient conditions for linear stability, in the (Re,δ)-plane:

$$\mathrm{Re}<\frac{2y^*}{k_1B(1-y^*)^2}\left(\frac{\pi^2}{\delta(1-y^*)^2}+\delta\left[1+\frac{2y^*k_2}{1-y^*}\right]\right),$$
$$\sim\frac{\delta}{k_1}[1+k_2 2^{1/2}B^{1/2}]+\frac{B\pi^2}{2\delta}\quad\text{as}\quad B\to\infty.$$

Thus, although our bound $\mathrm{Re}_{\mathrm{lin1}}$ is valid for all wavelengths, for long wavelength perturbations (small δ), we can also find a stability bound that increases linearly with B. For short wavelengths (large δ), the linear stability bound increases like $B^{1/2}$.

C. A "Squire-like" comparison result

Returning now to (23) and neglecting the Bingham term, we have the inequality

$$C_i\langle|\mathbf{u}|^2\rangle\leqslant\mathcal{I}(\mathbf{u})-\mathcal{V}(\mathbf{u}). \tag{42}$$

We make the mapping from y to $\xi=(y-y^*)/(1-y^*)$ giving

$$C_i(1-y^*)\int_0^1|\mathbf{u}|^2\mathrm{d}\xi\leqslant\frac{B(1-y^*)}{2y^*}\int_0^1 2\xi(u_rv_r+u_iv_i)\mathrm{d}\xi-\frac{1}{\mathrm{Re}(1-y^*)}\int_0^1\left|\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\xi}\right|^2\mathrm{d}\xi-\frac{(1-y^*)}{\mathrm{Re}}\int_0^1(\alpha^2+\beta^2)|\mathbf{u}|^2\mathrm{d}\xi. \tag{43}$$

As remarked previously, the two yielded regions for this stability problem completely uncouple and may be treated separately. By virtue of the boundary conditions (19)–(21), if we apply a similar mapping to the yielded region $y\in[-y^*,-1]$, i.e., $\xi=(y+y^*)/(1-y^*)$, we construct a velocity function $\mathbf{u}(\xi)$ that is continuous at $\xi=0$, in fact $\mathbf{u}(\xi)\in[H_0^1(0,1)]^3$. Deriving the identical integral expressions over the second yielded region and summing leads to

$$C_i(1-y^*)\int_{-1}^1|\mathbf{u}|^2\mathrm{d}\xi\leqslant\frac{B(1-y^*)}{2y^*}\int_{-1}^1 2\xi(u_rv_r+u_iv_i)\mathrm{d}\xi-\frac{1}{\mathrm{Re}(1-y^*)}\times\int_{-1}^1\left|\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\xi}\right|^2\mathrm{d}\xi-\frac{(1-y^*)}{\mathrm{Re}}\int_{-1}^1(\alpha^2+\beta^2)|\mathbf{u}|^2\mathrm{d}\xi. \tag{44}$$

If we now consider the analogous linear stability bound for a Newtonian fluid (using the mean velocity as the velocity scale, rather than the peak centerline velocity), we would derive the identity

$$C_i\int_{-1}^1|\mathbf{u}|^2\mathrm{d}y=\frac{3}{2}\int_{-1}^1 2y(u_rv_r+u_iv_i)\mathrm{d}y-\frac{1}{\mathrm{Re}_n}\int_{-1}^1\left|\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}y}\right|^2+(\alpha_n^2+\beta_n^2)|\mathbf{u}|^2\mathrm{d}y, \tag{45}$$

where α_n, β_n are the wavenumbers for the Newtonian normal mode and Re_n denotes the analogous Newtonian fluid Reynolds number. We note that $B(1-y^*)/y^*\leqslant 3$ for all B (this is the ratio of centerline velocity to mean velocity). Therefore, we have the following result.

Lemma 1: Suppose that $\mathrm{Re}_n<\mathrm{Re}_{\mathrm{Newt}}(\alpha_n,\beta_n)$ *gives sufficient conditions for the right-hand side of (45) to be negative, where* $\mathrm{Re}_{\mathrm{Newt}}(\alpha,\beta)$ *is any bound derived from (45), using only the properties of functions* $\mathbf{u}:\mathbf{u}\in[H_0^1(0,1)]^3$, *that satisfy also the following divergence free condition*:

$$0=i[\alpha u+\beta w]+\frac{\mathrm{d}v}{\mathrm{d}y}.$$

Then it follows that the Bingham fluid flow will be linearly stable at $\alpha=\alpha_n/(1-y^*)$, $\beta=\beta_n/(1-y^*)$ *if*

$$\mathrm{Re}<\frac{\mathrm{Re}_{\mathrm{Newt}}(\alpha,\beta)}{1-y^*}\,\frac{3y^*}{B(1-y^*)}. \tag{46}$$

Thus, at a shorter wavelength the Bingham flow will be stable for larger Reynolds numbers than have been established for the Newtonian flow. In particular, if a bound is derived that is independent of the wavelengths (α_n,β_n), then the Bingham fluid flow will be stable at a lower Reynolds number. Although apparently strong, this result is actually quite weak (but consistent with the nature of this type of analysis). In particular, this result does not imply that the Bingham fluid flow is necessarily ''more stable'' than the analogous Newtonian fluid flow, since both these bounds are likely to be quite conservative. On the other hand, all available analytical and numerical results, as well as practical experience with visco-plastic fluids, indicates this to be the case. Our result is ''Squire-like'' only in that it derives from a mapping of the wavenumbers.

IV. DISCUSSION AND CONCLUSIONS

The existing computational results on linear stability[13] show an almost linear increase in critical Reynolds number with B. On the other hand, the fully nonlinear stability theory[15] gives an increase like $B^{1/2}$. Hence, it is to be expected that our bounds will lie between $O(B)$ and $O(B^{1/2})$. From this perspective our results are both fully consistent and satisfactory.

In obtaining a bound of form $O(B^{1/2})$ we are using simply the purely viscous dissipation term. The effect of the yield stress is felt in reducing the width of the yielded region $(1-y^*)$. We can see this by straightforwardly balancing purely viscous (Newtonian) terms with the inertial terms in (23), and finding the optimal wavenumber to minimize the right-hand side. We note that this *purely viscous* balance is also the *balance* that is used in Ref. 15 for deriving the nonlinear stability bounds.

Our bound of $O(B^{3/4})$ has come from using the Bingham term in order to influence the *maximal* wavenumber $\delta_{\max}$, but having found this maximal wavenumber, it is still a viscous-inertial balance that gives the bound. We note that our most unstable wavenumber, $\delta_{\max}$ behaves like $\delta_{\max}=O(\mathrm{Re}/B^{1/2})$ as $B\to\infty$, i.e., $\delta_{\max}=O(B^{1/4})$ for our marginal bound, indicating that the short wavelengths cause most problems. Indeed for any bounded wavenumbers we can find a bound that is linear in B, but this still uses a viscous-inertial balance. This indicates that our bounds involve something additional to those for the nonlinear theory.

It is interesting to note that it is not the fact of having short wavelengths that is problematic, but rather the growth rate of $\delta=\delta_{\max}$ with B. For example, if the most unstable wavenumbers vary like $\delta\sim O(B^{\nu})$ for $\nu>1/4$, inspection of (32)–(34) reveals that the dominant dissipative term is in fact the Bingham term, and we can derive a bound of form Re $\lesssim O(B^{\nu+1/2})$. Thus, if $\nu>1/2$ it is even possible to find bounds that exceed the linear dependence with B. Since this would contradict the results in Ref. 13, we conclude that the most unstable wavenumber cannot grow faster than $O(B^{1/2})$. The variation of the wavenumber $\delta_{\max}$, at $\mathrm{Re}=\mathrm{Re}_{\mathrm{lin1}}$, is plotted in Fig. 4 [curve (1)]. Also plotted in Fig. 4 [curve (2)] is the computed wavenumber at the minimal critical Reynolds numbers for two-dimensional disturbances, as computed in Ref. 13. First, it is clear from Fig. 4 that we do in fact have a larger growth rate in the critical wavenumber for two-dimensional disturbances, than in $\delta_{\max}$, i.e., this is the source of conservatism in our bound. Second, the near linear dependence of the critical Reynolds number in Ref. 13 corresponds to $\nu\lesssim 1/2$ as $B\to\infty$, as can be observed in Fig. 4.

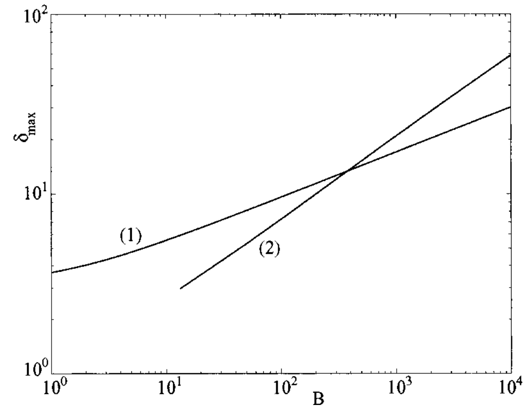

FIG. 4. (1) $\delta_{\max}$ versus B, at the critical Reynolds number $\mathrm{Re}=\mathrm{Re}_{\mathrm{lin1}}$; (2) axial wavenumber at the minimal critical Reynolds number for two-dimensional disturbances, see Ref. 13.

Since we cannot achieve a bound, uniform over all wavenumbers, that grows linearly with B, our results fall slightly short of confirming analytically the numerical results of Ref. 13, although we have explained above the results of Ref. 13 in the context of the growth rate of the critical wavenumbers with B. Of course, the real (quantitative) difference is that although we do have a linear dependence on the Bingham number in our inequalities, we still bound our integral expressions using quite general properties of functions, i.e., rather than requiring our functions to be eigenfunctions of the appropriate eigenvalue problem. In Ref. 13 the Bingham number appears directly in the Orr–Sommerfeld equation, and hence can directly influence the solution.

We might ask if an $O(B)$ uniform bound could be achieved for a simpler problem, e.g., a two-dimensional instability. Setting either $u=0$ or $v=0$ results in an unconditionally linearly stable problem. Setting $w=0$ returns us to the Orr–Sommerfeld problem of Ref. 13. Using the same method as above, i.e., bounding integral inequalities, we have not been able to improve to $O(B)$. With similar treatments, the Orr–Sommerfeld problem leads again to an $O(B^{3/4})$ uniform bound. Although the constant can certainly be improved over our estimate (41), only two-dimensional perturbations are covered. We conclude that $O(B^{3/4})$ is probably the best possible with this method. Perhaps a less rigorous method could lead to an $O(B)$ bound, and perhaps also to a much sharper estimate that could be used to guide computations. Here we have in mind the asymptotic method of Lin,[33] but have not pursued this approach so far.

Throughout we have focused on the Bingham number dependence, rather than that of the wavenumbers, and we have not directly considered the limit $B \to 0$. A partial explanation for this focus comes from Lemma 1, i.e., via a transformation we can extend existing results for the Newtonian stability problem to the Bingham fluid problem. Thus, any of the results in Refs. 16–18 can be re-cast as bounds for the Bingham fluid. However, these bounds will not take into account the Bingham number term in (23) and hence cannot give a stronger dependence on B. In reverse, we may also consider the limit $B \to 0$ in any of our bounds. To evaluate this limit, we make use of

$$y^*(B) \sim \frac{B}{3} - \frac{B^2}{6} \quad \text{as} \quad B \to 0,$$

and see for example that our uniform bound becomes

$$\mathrm{Re} < \mathrm{Re}_{\mathrm{lin1}} \sim \frac{4\pi}{3k_1}. \tag{47}$$

This is obviously quite weak, but is simply a reflection of the analysis being focused on B-dependence rather than on optimality for one fixed value of B. We also remark that the results in Refs. 16–18 are two-dimensional (due to Squire's theorem) and hence not directly comparable in any case. Undoubtedly, better Newtonian results can be found for three-dimensional Newtonian disturbances.

Finally, we remark that similar results can be derived for the Hagen–Poiseuille flow of a Bingham fluid (and presumably for other duct geometries). However, since the corresponding Newtonian flow is believed to be linearly stable, this arguably has less interest. Our wavenumber independent bound (in the limit that $B \to \infty$) is

$$\mathrm{Re} \lesssim \left[\frac{r^* \pi^2}{\sqrt{2}} \, \frac{\pi - 3}{28 + 20\pi} \right]^{1/2} B^{3/4}, \tag{48}$$

where r^* denotes the radial position of the yield surface for the basic flow (also a function of B, see, e.g., Ref. 15). As with the plane channel flow, a linear stability bound of form $\mathrm{Re} = O(B)$ can be straightforwardly derived if we allow bounds on the axial and azimuthal wavenumbers, α and n. The details for (48) are more involved but the ideas are similar. An interested reader may contact the authors.

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