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Predicting Transition to Turbulence in Well Construction Flows

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Abstract

Two classical geometries for wellbore hydraulics are the pipe and the plane channel, (often approximating an annulus). The technical literature is full of semi-empirical methodologies for computing frictional pressure losses and flow regimes in these geometries and there is much confusion about both the validity and physical basis of many of these phenomenological criteria.

In this paper we examine phenomenological criteria for transition, in the context of newly derived theoretical results on stability of these flows. We examine five transitional criteria commonly used in the petroleum industry and are able to rigorously demonstrate that only one of the five criteria can possibly be correct over the full range of Bingham numbers. We then examine a more limited, but relevant, range of Bingham numbers and show that enormous discrepancies exist between these criteria. We conclude by discussing implications for current practice.

Introduction

Shear flows of visco-plastic fluids are very common in well-construction flows. For example, in primary cementing and coiled tubing drilling, wellbore fluids flow in stationary pipes and eccentric annuli. In conventional drilling the pipe walls rotate and in operations such as tripping there is a relative axial motion of the pipe walls. In most of these operations, both laminar and turbulent flows are encountered, depending largely on operational conditions. Although frequently it is the static pressure that is responsible for maintaining primary control of the well, in certain situations frictional pressure may have an important, or even dominant, role. A particular case where this occurs is in extended reach drilling of near-horizontal wells, where frictional pressure is eventually responsible for limiting the length of the well.

In all applications listed above, accurate computation of wellbore hydraulics has become an important part of the well design. Various empirical correlations for the frictional pressure drop are employed in the industry, usually coded into computational software. A key feature of such hydraulics simulators is correct prediction of the flow regime. Without this prediction, a wide range of flows will undoubtedly be incorrectly classified, leading to inaccurate prediction of wellbore pressures.

In ths paper we present recent results on the problem of transition. We consider Poiseuille flow in both pipes and slots, i.e. a plane channel. The latter is often used in an oilfield setting as an approximation to the flow in a narrow annulus. Our focus is on the Bingham fluid model, which is commonly employed in the oilfield to characterise both drilling muds and cement slurries.

The approach we take is theoretical, in that we attempt to predict transition via the study of hydrodynamic instabilities. We mention at the outset that, even for the flow of a Newtonian fluid, theoretical predictions have not managed to yield an exact prediction of experimental values in these flow configurations. The classical approach adopted for plane Poiseuille flows of a Newtonian fluid has been to consider linear stability first and then develop weakly nonlinear methods based on the linear theory. For this configuration Squire’s theorem holds, (Ref. 1), and the linear problem is a two-dimensional one. Accurate numerical solutions of this problem, carried out in Ref. 2, predicts a Reynolds number $\mathrm{Re}_N > \mathrm{Re}_{N,lin} \approx 5772$ for the first linear instability to occur, where the Newtonian Reynolds number Re_N is here based on the channel half-width and maximum velocity. Weakly nonlinear stability theories have indicated that instability can result from sufficiently large amplitude finite perturbations for Reynolds numbers above ≈ 2900, (see e.g. Refs. 3-4). Energy stability methods consider fully nonlinear perturbations. These were applied to Poiseuille flow of Newtonian fluid in channels, pipes and annuli in the late 1960's, (Refs. 5-6), and are explained fully in Ref. 7. In contrast to the linear theory, these produce upper bounds for stability. For plane channel flow Ref. 5 predicts stability for $\mathrm{Re}_N < \mathrm{Re}_{N,E} \approx 99.207$. For the Hagen-Poiseuille flow (pipe flow) of a Newtonian fluid, there is no known critical Reynolds number for linear instability and the flow is believed to be linearly stable; see Refs. 8-11. Energy stability bounds are derived in Ref. 6, and predict stability for $\mathrm{Re}_N < \mathrm{Re}_{N,E} \approx 81.49$. Comprehensive surveys of the hydrodynamic stability theory for Newtonian fluid flows can be found Ref. 12, or the recent Ref. 13.

One key point to note is that experimentally obtained values for transition are an order of magnitude higher than the energy stability bounds, and are also significantly lower than the linear stability bounds, in both plane channel and pipe flow. This discrepancy between theoretical and experimental values is still not fully resolved, although partial explanations are offered in Refs. 14-15, and more recently the topic has again received attention, Refs. 13, 16-18.

Given the above background, it is not surprising that there is no definitive theoretical answer to the problem of predicting transition for a Bingham fluid. Here the basic Poiseuille flows are parameterized by the Bingham number, B, which relates directly to the width of the plug region. The key question therefore is how the transitional Reynolds number varies with increasing Bingham number.

For industrial application, this type of question is answered by a number of phenomenological theories. These began to appear in the literature in the 1950's and are widely used. A general approach is to form a parametric ratio of various physical flow quantities that are supposed to determine the stability of the flow. The value of this parametric ratio at which the Newtonian flow becomes unstable is known or calculated. It is then assumed that the same value of this parametric ratio will govern stability for analogous flows of all purely viscous non-Newtonian fluids. There are numerous examples of such criteria. Here we restrict our attention to those of Refs. 19-27. Although other phenomenological theories do periodically appear in the petroleum engineering literature, at their base they are usually drawn from one of Refs. 19-25, (and very often Ref. 19).

Insofar as hydrodynamic stability theory goes, the principal difficulty in simply generalizing the Newtonian results stems from having an unyielded plug region in the cente of the duct. In recent years, a better understanding of how to analyse these problems evolved. The linear theory has been developed in Refs. 28-29. For technical reasons, the extension of the linear theory to weakly nonlinear perturbations is not possible. The nonlinear theory is developed in Ref. 30. By comparison with the vast literature for the Newtonian case, the hydrodynamic theory is under-developed. However, inspite of this underdevelopment, the current results on hydrodynamic stablity, (Refs. 28-30), have important consequences for the phenomenological theories currently employed in the industry. A clear explanation of these consequences is the topic of this paper.

Basic flows

The Poiseuille flow solution for flow of a Bingham fluid in a plane channel or pipe, is easily found. The solutions have an unyielded plug region in the center of the flow, surrounded by shear layers. The dimensionless solutions are:

$$W(y)=\begin{cases}\dfrac{B}{2y^*}\left(1-y^*\right)^2 & |y|\le y^* \\ \dfrac{B}{2y^*}\left[\left(1-y^*\right)^2-\left(|y|-y^*\right)^2\right] & |y|>y^*\end{cases} \qquad (1)$$

and

$$W(r)=\begin{cases}\dfrac{B}{2r^*}\left(1-r^*\right)^2 & r\le r^* \\ \dfrac{B}{2r^*}\left[\left(1-r^*\right)^2-\left(r-r^*\right)^2\right] & r>r^*\end{cases} \qquad (2)$$

in plane channel & pipe geometries, respectively. In Eqns. 1&2, W is the axial velocity, with coordinates y and r measuring distance across the duct. The parameters y^* and r^* denote the yield surface positions in the plane channel and pipe, respectively. These parameters are uniquely determined by the Bingham number B, which is defined as:

$$B=\frac{\hat{\tau}_y\hat{R}}{\hat{\mu}\hat{U}_0} \qquad (3)$$

where $\hat{\tau}_y$ and $\hat{\mu}$, are the yield stress and plastic viscosity of the fluid, respectively. The length-scale $\hat{R}$ denotes the channel half-width, in the case of the plane channel, and the pipe radius in the pipe flow. The velocity scale, $\hat{U}_0$, is taken to be the mean velocity, (i.e. the volumetric flow rate divided by the cross-sectional area).

Since we have scaled with the mean velocity, there is an integral constraint placed on W, which leads to the following equations determining y^* and r^*:

$$0=\left(y^*\right)^3-3y^*\left[1+\frac{2}{\mathrm{B}}\right]+2 \qquad (4)$$

$$0=\left(r^*\right)^4-4\mathrm{r}^*\left[1+\frac{3}{\mathrm{B}}\right]+3 \qquad (5)$$

The parameters y^* and r^* are the positive real roots of the above polynomials that are less than unity, which are uniquely defined for positive B.

Typically, one solves for y^* and r^* numerically, see Figure 1. However, later we shall want to investigate the behaviour of y^* and r^* for large values of B. The asymptotic behaviour as $B\to\infty$ is:

$$y^*\sim 1-\frac{\sqrt{2}}{\sqrt{B}}+\frac{2}{3B}+O\left(B^{-3/2}\right) \qquad (6)$$

$$r^*\sim 1-\frac{\sqrt{2}}{\sqrt{B}}+\frac{1}{3B}+O\left(B^{-3/2}\right) \qquad (7)$$

and for completeness, as $B\to 0$ we have:

$$y^*\sim\frac{B}{3}-\frac{B^2}{6}+O\left(B^3\right) \qquad (8)$$

$$r^*\sim\frac{B}{4}+\frac{B^2}{12}+O\left(B^3\right) \qquad (9)$$

The asymptotic solutions are also plotted in Figure 1.

Note that, as with the Newtonian case, the Reynolds number does not enter into the definition of the basic flow.

However, the Reynolds number does enter into the various stability problems that can be considered. There are many definitions of the Reynolds number for a non-Newtonian fluid, and indeed it is this which makes comparison of different results rather tedious. Here we use throughout the paper the Reynolds number defined as:

$$\mathrm{Re} = \frac{\hat{\rho}\hat{R}\hat{U}_0}{\hat{\mu}} \tag{10}$$

i.e. straightforwardly using the mean velocity and the duct half-width (or radius).

Linear stability

Linear stability results have been computed numerically in Ref. 28, as the solution of an Orr-Sommerfeld problem. These are two-dimensional instabilities, rather than three-dimensional. For general linear perturbations there is no real equivalent to Squire's theorem, (but see Ref. 31 for such a result for a restricted set of perturbations). Therefore, it is possible that there are linear instabilities at lower Reynolds numbers than are predicted in Ref. 28. Since however, the role of the linear stability results is to provide an upper bound on the Reynolds number, above which instability is assured, the two-dimensional results are perfectly adequate.

Figure 2 shows the computed critical Reynolds number Re_{lin}, for two-dimensional linear instability, expressed as a function of B. It appears that Re_{lin} increases approximately linearly with B for large B. By numerically differentiating the data points, we can estimate the gradient of $\mathrm{Re}_{lin}(B)$ at large B, (i.e. up to the limit to which we have computed, $B \approx 8\times 10^6$, giving:

$$\begin{aligned} &\frac{d}{dB}\mathrm{Re}_{lin} \approx 36.9 \quad \text{at} \quad B \approx 8\times 10^6 \\ &\frac{d^2}{dB^2}\mathrm{Re}_{lin} \approx 3.5\times 10^{-7} \quad \text{at} \quad B \approx 8\times 10^6 \end{aligned} \tag{11}$$

Nonlinear stability

This situation is analysed in depth in Ref. 30. For arbitrary amplitude nonlinear perturbations, the flow will be stable provided that:

$$\mathrm{Re} < \mathrm{Re}_{GB} = \begin{cases} \dfrac{2y^*}{B\left(1-y^*\right)^3} & \text{Plane channel} \\[2ex] \dfrac{4\left(r^*\right)^2}{B\left(1+r^*\right)\left(1-r^*\right)^3} & \text{Pipe} \end{cases} \tag{12}$$

These bounds are obviously extremely conservative, but do give a rate of increase with B, and it is this that we are primarily concerned with. As $B \to \infty$ we have:

$$\mathrm{Re} < \mathrm{Re}_{GB} \sim \begin{cases} \sqrt{\dfrac{B}{2}} & \text{Plane channel} \\[2ex] \sqrt{\dfrac{B}{2}} & \text{Pipe} \end{cases} \tag{13}$$

However, in studying transition experimentally, one usually attempts to keep the flow laminar for as long as possible by suppressing perturbations as far as possible. Thus, transition occurs with perturbations that are usually of the order of the mean flow, or less, in an experimental setting. In Ref. 30 it is shown how conditional bounds may be derived for this type of perturbation. The results are that the flow is conditionally stable for Reynolds numbers $\mathrm{Re} < \mathrm{Re}_{CB}$, where for the plane channel:

$$\mathrm{Re}_{CB} = \frac{\dfrac{2y^*\,\mathrm{Re}_{Busse}}{B\left(1-y^*\right)^3}}{\left[1+\dfrac{\sqrt{2}y^*}{3\,\mathrm{Re}_{Busse}}\right]^2\left[1+\dfrac{\sqrt{2}y^*}{3\,\mathrm{Re}_{Busse}}+\dfrac{\sqrt{2}y^*}{3}\right]} \tag{14}$$

and for the pipe flow:

$$\mathrm{Re}_{CB} = \frac{\dfrac{2r^*\,\mathrm{Re}_{Joseph}}{B\left(1-r^*\right)^3}}{\left[1+\dfrac{\sqrt{2}r^*}{3\,\mathrm{Re}_{Joseph}}\right]^2\left[1+\dfrac{\sqrt{2}r^*\left(1+r_2\right)}{6r_2}+\dfrac{\sqrt{2}r^*}{3\,\mathrm{Re}_{Joseph}}\right]} \tag{15}$$

where $\mathrm{Re}_{Busse} = 99.207$ and $\mathrm{Re}_{Joseph} = 81.49$, are constants computed in Refs. 5 & 6, respectively. As $B \to \infty$ we have the following limits:

$$\mathrm{Re}_{CB} \sim \frac{\mathrm{Re}_{Busse}\sqrt{B}}{2\left[1+\dfrac{\sqrt{2}}{3\,\mathrm{Re}_{Busse}}\right]^2\left[1+\dfrac{\sqrt{2}}{3\,\mathrm{Re}_{Busse}}+\dfrac{\sqrt{2}}{3}\right]} \tag{16}$$

for the plane channel flow and

$$\mathrm{Re}_{CB} \sim \frac{\mathrm{Re}_{Joseph}\sqrt{B}}{2\left[1+\dfrac{\sqrt{2}}{3\,\mathrm{Re}_{Joseph}}\right]^2\left[1+\dfrac{\sqrt{2}}{3}+\dfrac{\sqrt{2}}{3\,\mathrm{Re}_{Joseph}}\right]} \tag{17}$$

for the pipe flow. It is the conditional bounds that we shall use most in the following sections.

Phenomenological criteria

Laminar and turbulent flows of slurries and suspensions with Bingham fluid-like rheological behaviour are used for a number of hydraulic applications in the petroleum industry. Consequently, it is hardly surprising that a number of phenomenological theories have been developed over the years. The chief purpose of these theories has been to quantify the frictional pressure drop in different geometries and flow situations. However, these theories often also contain criteria that predict transition between laminar and turbulent regimes. We summarise below the principal approaches taken.

Metzner and Reed

Metzner and Reed (Ref. 19) used the Fanning friction factor F as their stability parameter. A range of data from pipe flow experiments with different non-Newtonian fluids indicated that the data deviated from the laminar flow curve, at approximately the same ratio of viscous shear to inertial forces as do Newtonian fluid data in smooth pipes, i.e. $F \approx 0.008$. They therefore proposed that for all time independent non-

Newtonian fluids, transition would take place when F drops to a value of about $F \approx 0.008$, or less. Here, we have fixed the critical value at $F = F_c = 0.076$, corresponding to a critical Reynolds number for the Newtonian flow of $\mathrm{Re}_c = 2100$.

The Metzner-Reed criterion is commonly applied in geometries other than circular pipes. For such geometries a modified Reynolds number can be defined in such a way that in laminar flow, the friction factor relation is identical to that obtained for the pipe flow. Experimental studies using Newtonian fluid flowing through rectangular ducts of different aspect ratios, (Ref. 32), and through annular ducts of different radius ratio, (Ref. 33), show results broadly similar to the pipe flow, for both laminar and turbulent flows. In particular, the transition occurs at a modified Reynolds number of ≈ 2000, that is $F \approx 0.008$. The Reynolds number defined by Metzner and Reed has thus been generalised, (Ref. 34), to apply to laminar flows of purely viscous non-Newtonian fluids through ducts of arbitrary cross section. Experimental results (Refs. 35-36), for power law fluids flowing in rectangular ducts, demonstrate the utility of the modified Reynolds number as a hydraulic parameter and that the transition occurs at $F \approx 0.008$. The straightforward generalisation of Metzner and Reed's criterion is therefore to assume that for all purely viscous non-Newtonian fluids, transition would take place at $F = F_c = 0.076$, independent of the duct geometry.

Hedström

According to Hedström, (Ref. 20), for given Hedström number, $He = \mathrm{Re}\,B$, transition occurs at the point of intersection of the laminar and turbulent friction factor curves. Hedström argues that for a fully turbulent fluid the effect of the yield stress is negligible, and therefore uses the well known Nikuradse relation for the turbulent friction factor in the pipe. For the plane channel flow, based on the results of Refs. 32-33, a similar Nikuradse relation is assumed for the turbulent friction factor, but using a modified Reynolds number.

Local Stabilty: Ryan & Johnson, Hanks

Two identical predictions of transitional Reynolds numbers have been made by Ryan & Johnson, (Ref. 21), and by Hanks, (Refs. 22-24). What is interesting is that these two predictions are arrived at using quite different approaches. Ryan & Johnson suggested using the ratio of input energy to energy dissipation for an element of fluid as the stability parameter. They examine the situations in which the energy of a disturbance increases or decreases with time, considering the energy equation for a linear two-dimensional disturbance. The rate of increase of kinetic energy is equal to the difference between the rate at which energy is converted from the basic flow to the disturbance, via a Reynolds shear stress term. A ratio ζ is formed between the rate of increase of kinetic energy and the rate at which energy is dissipated. The ratio ζ varies with radial position r and thus their approach can be thought of as a local approach.

It is assumed that transitional instabilities will first appear at the radial position where ζ is maximal, for all purely viscous non-Newtonian fluids. Further, it is assumed that instability occurs when this maximal value of ζ exceeds a critical number, ζ_{crit}, regardless of the exact fluid type. The critical number ζ_{crit} is defined from the transition of a Newtonian flow and is given by $\zeta_{crit} = 808$.

Hanks derives exactly the same criterion for transition as Ryan & Johnson. However, his reasoning is different and is in some way more direct. Hanks identifies the key mechanism leading to transitional instability as being the rotational momentum transfer. Hanks introduces a parameter, ζ_H, which is almost identical to that of Ryan and Johnson. However, for Hanks ζ_H represents a balance between the rate of change of angular momentum of a deforming fluid element and its rate of loss of momentum due to frictional drag. Hanks' ratio ζ_H is exactly twice that of Ryan and Johnson. Hanks also assumes that his transition criterion does not depend on the constitutive equation, for purely viscous fluids, or on the geometry of flow. The critical value of ζ_H, where the laminar motion becomes unstable, $\zeta_{H.crit}$, was determined to be $\zeta_{H.crit} = 404$, when the theory was applied to axial isothermal Newtonian flow in a pipe. The plane channel flow is considered, by analogous methods, in Ref. 23.

Integral Stability: Mishra & Tripathi

Mishra & Tripathi (Ref. 25) postulate that the important factors governing transition to turbulence are the mean kinetic energy and the wall shear stress. Furthermore, the onset of turbulence will occur at the same critical ratio of these quantities for all purely viscous non-Newtonian fluids. This ratio is fitted from its value for Newtonian fluid transition in a pipe. For the plane channel flow, we have simply generalised this approach using a modified Reynolds number.

Slatter's approach

A recent phenomenological approach is that due to Slatter (Refs. 26-27). He treats the unyielded region as a solid body, that has no effect on the stability of flow. Therefore, only the flow of the sheared fluid in the *annulus*, between pipe wall and unyielded plug, is considered. Slatter defines a Reynolds number based on the effective viscosity of the annular flow, i.e. using the hydraulic diameter of the annulus and an effective rate of strain. Although Slatter has not considered a plane channel flow, the phenomenological reasoning is easily extended to the plane channel.

Variation of Re_c with B

In the above descriptions of the phenomenological criteria, we have purposefully omitted algebraic details. The different criteria are formulated and expressed with a variety of different notations and dimensionless parameters, which makes direct comparison tedious. To ease the task of making a direct comparison, we have converted each of the criteria into our own dimensionless parameters Re and B. These are listed below.

Metzner & Reed (Ref. 19)

$$\mathrm{Re}_c = \begin{cases} \dfrac{262.5B}{y^*} & \text{Plane channel} \\ \dfrac{262.5B}{r^*} & \text{Pipe} \end{cases} \tag{18}$$

Ryan & Johnson (Ref. 21); Hanks (Refs. 22-24)

$$\mathrm{Re}_c = \begin{cases} \dfrac{2100y^*}{B(1-y^*)^3} & \text{Plane channel} \\ \dfrac{4200r^*}{B(1-r^*)^3} & \text{Pipe} \end{cases} \tag{19}$$

Hedström (Ref. 20)

$$\mathrm{Re}_c = \text{Computed numerically from intersection}$$

Mishra & Tripathi (Ref. 25)

$$\mathrm{Re}_c = \begin{cases} \dfrac{405B}{y^*C} & \text{Plane channel} \\ \dfrac{262.5B}{r^*C} & \text{Pipe} \end{cases} \tag{20}$$

Slatter (Ref. 26)

$$\mathrm{Re}_c = \begin{cases} \dfrac{262.5(B+12W_s/D_s)}{W_s^2} & \text{Plane channel} \\ \dfrac{262.5(B+8W_{ann}/D_{ann})}{W_{ann}^2} & \text{Pipe} \end{cases} \tag{21}$$

In the above, Mishra & Tripathi's parameters are:

$$C = \begin{cases} \dfrac{B^3(1-y^*)^6}{8y^*}\left[1+\dfrac{16(1-y^*)}{35y^*}\right] & \text{Plane channel} \\ \dfrac{B^3(1-r^*)^6}{8r^*}\left[\dfrac{1}{2}+\dfrac{1-r^*}{(r^*)^2}\left(\dfrac{(1-r^*)}{8}+\dfrac{16r^*}{35}\right)\right] & \text{Pipe} \end{cases} \tag{22}$$

For Slatter, D_{ann} is the dimensionless hydraulic diameter of the annulus, $D_s = 4(1-y^*)$, and the other parameters are:

$$W_s = \frac{B(1-y^*)^2}{3y^*}$$
$$W_{ann} = \frac{B(1-r^*)^2}{r^*(1+r^*)}\left[\frac{2}{3}r^* + \frac{1-r^*}{4}\right] \tag{23}$$

Asymptotic limit $B \to \infty$

Many well construction fluids have a significant yield stress, so that the limit of the criteria as $B \to \infty$ is clearly of interest. We use here the asymptotic expressions, Eqns. 6&7, to determine the limting forms of the above criteria.

Metzner & Reed (Ref. 19)

$$\mathrm{Re}_c \sim \begin{cases} 262.5B + O(B^{1/2}) & \text{Plane channel} \\ 262.5B + O(B^{1/2}) & \text{Pipe} \end{cases} \tag{24}$$

Ryan & Johnson (Ref. 21); Hanks (Refs. 22-24)

$$\mathrm{Re}_c \sim \begin{cases} 1050\sqrt{\dfrac{B}{2}} + O(1) & \text{Plane channel} \\ 2100\sqrt{\dfrac{B}{2}} + O(1) & \text{Pipe} \end{cases} \tag{25}$$

Hedström (Ref. 20)

$$\mathrm{Re}_c = \text{Computed numerically from intersection}$$

Mishra & Tripathi (Ref. 25)

$$\mathrm{Re}_c = \begin{cases} 405B + O(B^{1/2}) & \text{Plane channel} \\ 525B + O(B^{1/2}) & \text{Pipe} \end{cases} \tag{26}$$

Slatter (Ref. 26)

$$\mathrm{Re}_c = \begin{cases} 590.63B + O(B^{1/2}) & \text{Plane channel} \\ 590.63B + O(B^{1/2}) & \text{Pipe} \end{cases} \tag{27}$$

Comparison with results from Refs. 28-30

Here we provide a comparison of the different predictions made of transition. The results are slightly different in the case of the plane channel and the pipe, since in the case of the plane channel we have predictions of Re_{lin}. Our comparison between theoretical and phenomenological criteria is also not direct. The phenomenological criteria are predictions of a transitional Reynolds number, whereas linear and nonlinear stability criteria are upper and lower bounds for transitional instabilities, i.e. above the linear stability bound instabilities do occur, below the nonlinear bound they do not occur. As the nonlinear bound, we present Re_{CB}. As explained, this is a conditional bound for stability to perturbations that are of the same size as the mean flow. This is consistent with what one might expect in an experimental study of transition, i.e. transition typically occurs with a small finite perturbation.

In Figure 3 we show the variations of the critical Reynolds number for transition, Re_c, as a function of the Bingham number B for the plane channel. Curve 7 in Figure 3 shows Re_{lin}, the lowest known value for which the flow is linearly unstable. Curve 6 in Figure 3 shows the conditional bound for nonlinear stability, Re_{CB}. Figure 4 shows the same comparison for the pipe flow results. A number of observations can be made, concerning the results presented in Figures 3 & 4.

(i) All criteria indicate an increase of Re_c with increasing B.

(ii) For all B, the phenomenological predictions Re_c are ranged in the same way for the two flows:

$$\mathrm{Re}_{c,Hanks} < \mathrm{Re}_{c,Metzner} < \mathrm{Re}_{c,Mishra} < \mathrm{Re}_{c,Slatter} \tag{28}$$

For sufficiently large B, Hedstrom's prediction also exceeds that of Slatter.

(iii) For the plane channel, for $B > 200$, only Hanks criterion (Refs. 21-24) lies between the two theoretical limits. If we consider the asymptotic behaviours above, (Eqns. 11, 16-17, 24-27), it is observed that each of the other limits increases linearly with B and at a much greater rate than the linear stability bound. Unless there exists a value for B at which the transition ceases to be subcritical, (thought to be very unlikely), each of these criteria eventually fail to have any predictive validity.

(iv) In fact, for the plane channel flow, the asymptotic rate of increase of each of the phenomenological criteria, (except that of Refs. 21-24), is approximately one order of magnitude larger than the rate of increase of the linear stability bound Re_{lin}. The asymptotic expressions actually approximate the full expressions for Re_c quite well, even for relatively low values of B, e.g. the approximation is reasonable at $B = 10$ and is very close after $B = 30$. Thus, the range in B over which any of the phenomenological criteria, apart from that of (Refs. 21-24), is likely to accurately describe transition is actually fairly limited.

As we approach the limit $B \to 0$, the different phenomenological criteria are in agreement, since they are based on the same Newtonian value. A comparison of the different phenomenological criteria in the lower range of B is given in Figure 5. The Hedström curve is based on the Nikuradse relation, which is valid for $\mathrm{Re}_{Dh} \geq 4000$, (where Re_{Dh} is Reynolds number based on hydraulic diameter). Thus, to construct Figure 5 this curve has been extrapolated to the Newtonian value, as indicated by the dashed line. It should be noted that the critical Reynolds number based on the intersection method therefore assumes that the abrupt jump in frictional pressure loss (characteristic of Newtonian flow) is absent. In addition, the success of this method depends on the accuracy of the turbulent correlation used. In comparison with the other criteria, (leaving Hanks criterion aside), Figure 5 indicates that the Newtonian model (Nikuradse relation) tends to underestimate Re_c for low values of B, and overestimate Re_c for large B. Finally, an immediate observation from Figure 5 is that even relatively close to $B > 0$ the different phenomenological criteria diverge significantly.

Conclusions

Our results demonstrate conclusively that, for the plane channel, all phenomenological criteria except that of Hanks (equivalently Ryan & Johnson) become invalid at large B, i.e. since the linear stability bound is violated. Considering the pipe flow, we do not expect that the actual rate of increase of Re_c with B, will differ by an order of magnitude from the results in the plane channel, nor do we expect that the asymptotic rate of increase should be different. Therefore, we strongly expect that all the rates of increase that are asymptotically linear with B, (i.e. all except that of Hanks), are also invalid in the limit $B \to \infty$ for the pipe flow. These criteria simply increase too fast with B. These asymptotic rates of increase are approached rapidly, for $B > 10$; see in Figure 1 for both geometries. Although one might argue that these criteria are not incorrect until they violate the linear stability criterion, this is not a safe interpretation to make. The linear increase is too fast and this indicates that there is something fundamentally wrong with the underlying physical assumptions. The fact that these linear rates are achieved at relatively low Bingham numbers suggests that the criteria for transition should not be used at all.

Our results do not give a definitive indication of whether Hanks' criterion, or any of the others, gives a good prediction within the range where they do not violate the linear stability criterion. In the sense that neither theoretical limit is violated, all criteria can be considered equally valid within the most practical range of B, say $0 < B < 50$. The large differences between the criteria shown in Figure 5 is worrying. Probably the only way to determine which of the criteria is most applicable would be to compare directly with experimentally derived values.

Most of experimental studies of transition for yield stress fluids are based on frictional pressure loss measurements. As an example, a recent attempt to empirically characterize transition in large pipes has been made by Slatter and Wasp, (Ref. 27). In Ref. 27, three correlations are presented, to cover transitional Reynolds numbers over the full range of Hedström numbers. Translated into our parameters, these correlations and ranges are approximately:

$$\mathrm{Re}_c \approx \begin{cases} 1050B & 0 \leq B \leq 0.4047 \\ 1701.39B^{0.538} & 0.4047 < B \leq 27.126 \\ 676B & 27.126 < B \end{cases} \qquad (29)$$

Whilst the mid-range correlation is close to $\sqrt{B}$, the high range correlation is similar to those given earlier and will violate the linear stablity criterion. The data set in Ref. 27 extends only up to $B \approx 100$, and is in fact based on fluids that are first characterized as Herschel-Bulkley fluids and then the rheological behaviour is somehow extrapolated to the Bingham fluid model.

Studies such as Ref. 27 highlight some significant problems with both oilfield usage and with studying transition experimentally for a Bingham fluid. Firstly, few fluids that have a yield stress show a completely linear relationship between shear stress and rate of strain over a sufficiently wide range of shear rates to make the Bingham fluid characterisation reasonable for a study of transitional flows. Inevitably, any experimental study that is compared to Bingham fluid results requies some sort of extrapolation of the rheological data, as in Ref. 27. However, how to extrapolate in a way that preserves accurately the stability characteristics is completely unknown. In the oil industry the Bingham model is employed mostly for ease of use, i.e. in fitting field data. Yield points for wellbore fluids are nearly always an extrapolation. Although 3-parameter models such as the Herschel-Bulkley model provide a better “fit” to the data, these models are not universally used. At high shear rates, models such as the Herschel-Bulkley are also not really appropriate, since they predict that the effective viscosity vanishes, whereas in reality some sort of Newtonian plateau is usually attained.

A second problem is that the method used to identify transition in an industrial flow is often based on detecting the change in frictional pressure drop. For several non-Newtonian fluids (slurries, especially), the abrupt change in frictional pressure loss at transition is largely absent, (see for example Refs. 37-38). Thus, caution is needed in interpreting transition data for industrial slurry flows, e.g. are we seeing the start or finish of transition? A number of secondary local measurements should be made for comparison, but this is not always feasible in an industrial setting, and especially not always at the wellsite. Furthermore, in an oilfield setting, where duct geometries will vary slowly along the length of the flow path, transition is very likely to be localized, i.e. parts of the well are likely to be laminar, turbulent and transitional at the same time. We have found only three papers (Refs. 37,39-40) that deal in detail with properties of the flow and making of careful secondary measurements near to transition. In Refs. 37 & 39, the centerline velocity is measured with help of LDV system and detection of transition is based on the fact that for laminar pipe flows the centerline velocity is substantially larger than for turbulent flows. Ideally, the time trace of the axial velocity at different radial locations should be analysed, in order to detect the first appearance of the turbulence burst corresponding to the start of transition. This kind of analysis has not, to our knowledge, been done for the fluids with which we are concerned.

On the theoretical side, it is worth mentioning that the problem of predicting the onset of transition for a Bingham fluid is perfectly well defined mathematically. Our work in Refs. 28-30 is simply a starting point which, although incomplete, we have seen has serious implications. Given the difficulties in making experimental (or field) comparison with the Bingham model, as explained above, one might question whether it would be easier to derive analogous stability results for more complex yield stress models, such as the Herschel-Bulkley or Casson model. To our knowledge, this has not been done. We comment only that linear stability analyses are not straightforward for these nonlinear models and that the nonlinear analysis is also more challenging.

Finally, we must comment that there are a great many experimental studies, correlations and phenomenological expressions derived for the frictional pressure losses in turbulent flows of (visco-plastic) industrial suspensions and slurries. Our results have no direct bearing on these results, except in calling into question the prediction of the point at which transition occurs. For oilfield application, this is likely to become important only in planning of critical operations, such as extended reach drilling, where frictional pressure is the limiting factor.

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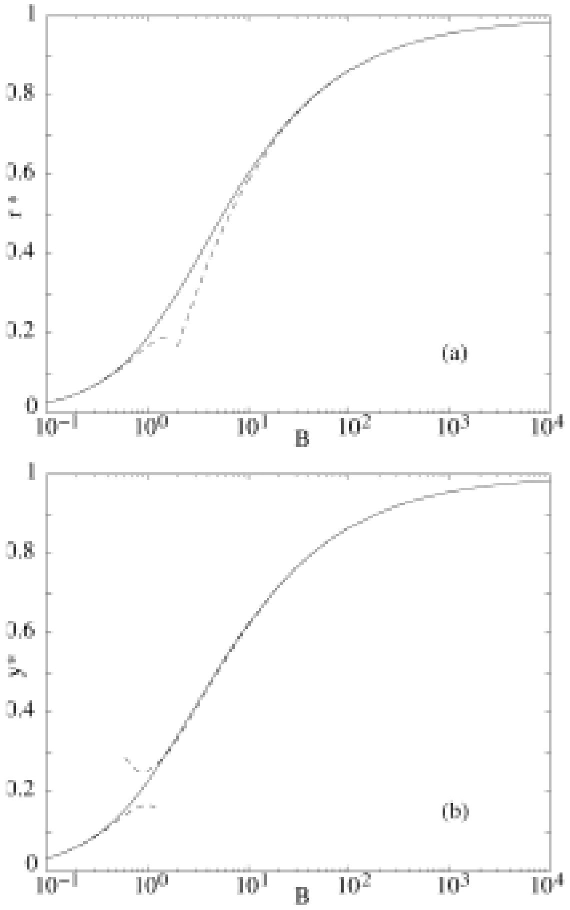

Figure 1: Variation in the dimensionless yield surface thickness: a) Pipe: $r^*(B)$; b) Plane channel $y^*(B)$. The dashed lines show asymptotic expansions to $r^*(B)$ and $y^*(B)$, valid for large and small B.

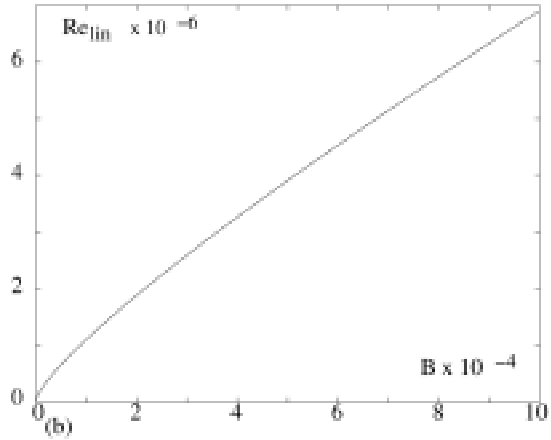

Figure 2: Computed critical Reynolds number Re_{lin}, for two-dimensional linear instability, expressed as a function of B, see Ref. 28.

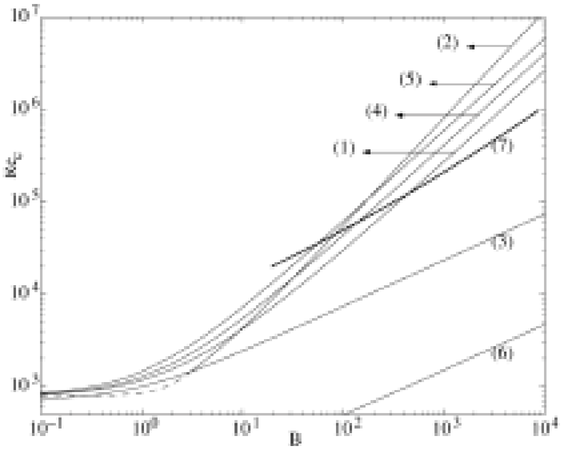

Figure 3: Critical Reynolds number as a function of a Bingham number for plane channel flow; comparison between the different criteria : (1) Metzner and Reed; (2) Hedström; (3) Hanks; (4) Integral stability; (5) Slatter; (6) Conditional stability; (7) Linear stability.

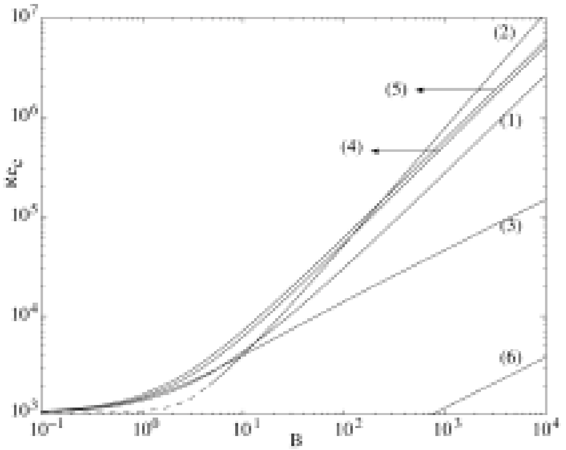

Figure 4: Critical Reynolds number as a function of a Bingham number for pipe flow; comparison between the different criteria : (1) Metzner and Reed; (2) Hedström; (3) Hanks; (4) Integral stability; (5) Slatter; (6) Conditional stability.

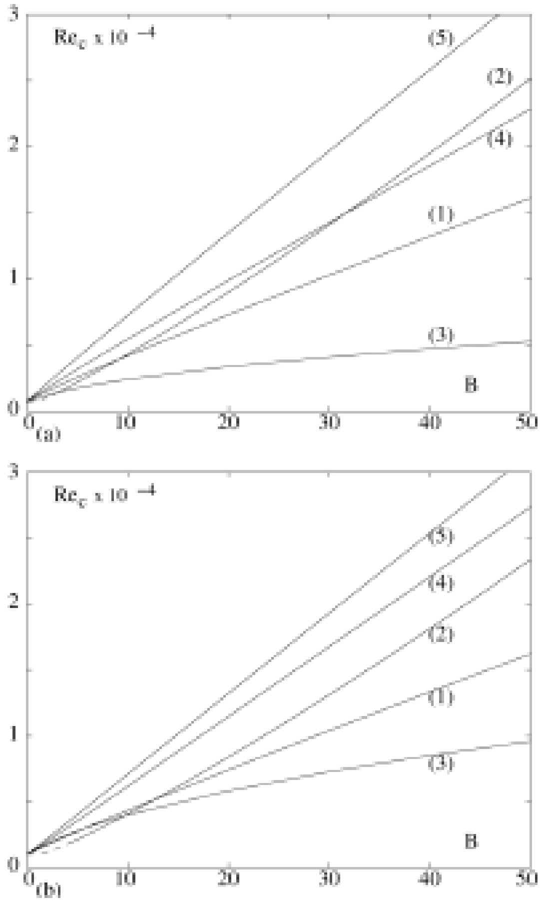

Figure 5. Comparison between the different phenomenological criteria, in the low Bingham number range: (a) plane channel flow, (b) pipe flow: (1) Metzner and Reed; (2) Hedstrom; (3) Hanks; (4) Integral stability; (5) Slatter.