The smallest singular value of random matrices with independent entries

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Abstract

We consider a classical problem of estimating the smallest singular value of random rectangular and square matrices with independent identically distributed entries. The novelty of our results lies in very weak, or nonexisting, moment assumptions on the distribution of the entries. We prove that, given a sufficiently “tall” $N \times n$ rectangular matrix $A = (a_{ij})$ with i.i.d. entries satisfying the condition $\sup_{\lambda \in \mathbb{R}} P\{|a_{ij} - \lambda| \leq 1\} \leq 1/2$, the smallest singular value $s_n(A)$ satisfies $s_n(A) \gtrsim \sqrt{N}$ with probability very close to one.

Our second theorem is an extension of the fundamental result of Bai and Yin from the early 1990’s. Let $\{a_{ij}\}_{i,j=1}^{\infty}$ be an infinite double array of i.i.d. random variables with zero mean and unit variance, and let $(N_m)_{m=1}^{\infty}$ be an integer sequence satisfying $\lim_{m \to \infty} \frac{N_m}{m} = r$ for some $r \in (1, \infty)$. Then, denoting by $A_m$ the $N_m \times m$ top-left corner of the array $\{a_{ij}\}$, we have

$$\lim_{m \to \infty} \frac{s_m(A_m)}{\sqrt{N_m}} = \sqrt{r} - 1 \text{ almost surely.}$$

This result does not require boundedness of any moments of $a_{ij}$’s higher than the 2-nd and resolves a long standing question regarding the weakest moment assumptions on the distribution of the entries sufficient for the convergence to hold.